Bounds for the Spectral Mean Value of Central Values of L-functions

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Abstract

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of L-functions at $s = \frac{1}{2}$

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We prove two results about the boundedness of spectral mean value of Rankin–Selberg L-functions at $s = \frac{1}{2}$, which is an analogue for Eisenstein series of X. Li's result for Hecke–Maass forms.

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Chapter 1

Introduction

1.1 Background

L-functions are fundamental objects in number theory which carry rich arithmetic information such as geometric invariants (for instance, the Birch and Swinnerton-Dyer conjecture for elliptic curves). Bounds for L-functions on the critical line ($\mathrm{Re}(s) = \frac{1}{2}$) are related to interesting problems like the problem of equidistribution of integer points on surfaces [18] and Hilbert's eleventh problem (i.e. which integers are integrally represented by a given quadratic form over a number field) [18,43].

Sharp upper bounds for central values for an individual L-function could be easily derived from the Riemann hypothesis, but strong unconditional bounds are much more difficult to obtain. Important works in this field for $GL(1)$ and $GL(2)$ L-functions have been done by Weyl [48], Burgess [5], Good [15], Meurman [35], Duke, Friedlander and Iwaniec [9–11], Sarnak [44], Kowalski, Michel and Vanderkam [26], Michel [36], Harcos and Michel [19], Michel and Venkatesh [37,38], Lau, Liu and Ye [31], to name a few. Much less is known for higher rank groups.

Conrey and Iwaniec developed a spectral method to find the currently best known unconditional bounds for $GL(1)$ L-functions in a landmark paper [6]. Instead of a single L function, they considered a family of L-functions and find the upper bound for their (weighted) mean value by harmonic analysis (Petersson–Kuznetsov trace formula). With a suitable choice of the weight, they proved that $L\left(\frac{1}{2},\chi\right) \ll q^{\frac{1}{6}+\varepsilon}$ for a Dirichlet L-function with character $\chi \pmod q$.

Recently X. Li found new methods to obtain exciting results on Rankin–Selberg convolutions of $GL(3) \times GL(2)$ Maass forms including non-vanishing of central values [33] and subconvexity bounds [34]. A remarkable technique in her work is the application of the Voronoi formula [13, 14] for $SL(3,\mathbb{Z})$ which was first discovered by S. D. Miller and W. Schimid [40, 41].

In this thesis we apply Li's method to a different situation that she suggested. Instead of the Rankin–Selberg convolution for Hecke–Maass forms, we consider that of a $GL(3)$ Eisenstein series and a spectral family of $GL(2)$ forms. In this case, the Rankin–Selberg convolutions split into third powers of L-functions for $GL(2)$ forms, or sixth powers of the shifted Riemann zeta function, or products of such L-functions. As a consequence we establish bounds for these L-functions, as well as obtain a new proof of [6] for the untwisted case with a better bound.

1.2 Main theorems

Theorem 1.2.1. *Let $\left\{u_j\right\}$ be an orthonormal basis of even Hecke–Maass forms for $SL(2,\mathbb{Z})$ corresponding to the Laplacian eigenvalue $\left(\frac{1}{4}+t_j^2\right)$ with $t_j \geq 0$. Then for $\varepsilon > 0$, large T and $T^{\frac{3}{8}+\varepsilon} \leq M \leq T^{\frac{1}{2}}$, we have*

$$\sum_j{}' e^{-\frac{(t_j-T)^2}{M^2}} \left|L\left(\tfrac{1}{2},u_j\right)\right|^3 + \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-\frac{(t-T)^2}{M^2}} \left|\zeta\left(\tfrac{1}{2}-it\right)\right|^6 \, dt \ll_\varepsilon T^{1+\varepsilon} M \tag{1.2.1}$$

where $'$ means summing over the orthonormal basis of even Hecke–Maass forms.

Corollary 1.2.2. $L\left(\frac{1}{2},u_j\right) \ll t_j^{\frac{11}{24}+\varepsilon}$.

Remarks. Ivić [21] proved the stronger bound $L\left(\frac{1}{2},u_j\right) \ll t_j^{\frac{1}{3}+\varepsilon}$ which is currently the world record.

In a paper by Iwaniec [23], he also proved $L\left(\frac{1}{2},u_j\right) \ll t_j^{\frac{1}{3}+t}$ but conditionally. This proof can now be made unconditional.

Corollary 1.2.3. $\left|\zeta\left(\frac{1}{2}+it\right)\right| \ll t^{\frac{11}{48}+\varepsilon}$.

Remark. We include the bound for $\zeta\left(\frac{1}{2}+it\right)$ only because it follows directly from the Main Theorem. Much stronger results are known. [20].

Theorem 1.2.4. *Let ψ be an even Hecke–Maass form for $SL(2,\mathbb{Z})$, $\left\{u_j\right\}_j$ an orthonormal basis of even Hecke–Maass forms for $SL(2,\mathbb{Z})$ corresponding to the Laplacian eigenvalue $\left(\frac{1}{4}+t_j^2\right)$ with*

$t_j \geq 0$. *Then for* $\varepsilon > 0$, *large* T *and* $T^{\frac{3}{8}+\varepsilon} \leq M \leq T^{\frac{1}{2}}$, *we have*

$$\sum_j{}' e^{-\frac{(t_j-T)^2}{M^2}} L\left(\tfrac{1}{2}, u_j\right) L\left(\tfrac{1}{2}, \psi \times u_j\right) + \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-\frac{(t-T)^2}{M^2}} \left|\zeta\left(\tfrac{1}{2}+it\right) L(\tfrac{1}{2}+it, \psi)\right|^2 dt \ll_\varepsilon T^{1+\varepsilon} M \tag{1.2.2}$$

where $'$ *means summing over the orthonormal basis of even Hecke–Maass forms.*

Corollary 1.2.5. $L\left(\frac{1}{2}, u_j\right) L\left(\frac{1}{2}, \psi \times u_j\right) \ll t_j^{\frac{11}{8}+\varepsilon}$.

Corollary 1.2.6. $\left|\zeta\left(\frac{1}{2}+it\right) L(\frac{1}{2}+it, \psi)\right| \ll t^{\frac{11}{16}+\varepsilon}$.

1.3 A review of the $GL(2)$ spectral decomposition

Since the proof of the main theorems is based on a spectral method, let us recall some standard facts about the spectral decomposition of $L^2(SL(2,\mathbb{Z})\backslash\mathfrak{h}^2)$.

Let $\mathfrak{h}^2 = \{ x+iy \in \mathbb{C} \mid x \in \mathbb{R}, y \in \mathbb{R}_+ \}$ be the classical upper half plane. The $SL(2,\mathbb{Z})$-invariant Laplace operator

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

has a spectral decomposition on $L^2\left(SL(2,\mathbb{Z})\backslash\mathfrak{h}^2\right)$ as follows:

$$L^2(SL(2,\mathbb{Z})\backslash\mathfrak{h}^2) = \mathbb{C} \oplus C\left(SL(2,\mathbb{Z})\backslash\mathfrak{h}^2\right) \oplus \mathcal{E}\left(SL(2,\mathbb{Z})\backslash\mathfrak{h}^2\right),$$

where $\mathbb{C}$ is the space of constant functions, $C\left(SL(2,\mathbb{Z})\backslash\mathfrak{h}^2\right)$ is the space of Hecke–Maass cusp forms and $\mathcal{E}\left(SL(2,\mathbb{Z})\backslash\mathfrak{h}^2\right)$ is the space of Eisenstein series.

Let $\mathcal{U} = \left\{u_j \;\middle|\; j \geq 1\right\}$ be an orthonormal basis for the space $C\left(SL(2,\mathbb{Z})\backslash\mathfrak{h}^2\right)$, where u_j's are Hecke–Maass cusp forms with Laplacian eigenvalues $\left(\frac{1}{4}+t_j^2\right)$ ($t_j \geq 0$) and with Hecke eigenvalues $\lambda_j(n)$.

Each $u_j(z)$ has the Fourier expansion

$$u_j(z) = \sum_{n\neq 0} \rho_j(n) W_{s_j}(nz)$$

where

$$W_s(z) = 2|y|^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|y|)e(x) \tag{1.3.1}$$

is the Whittaker function. Here $K_s(y)$ is the K-Bessel function, and $e(x) := e^{2\pi i x}$.

Furthermore, for $n > 0$, we have

$$\rho_j(\pm n) = \rho_j(\pm 1)\lambda_j(n)n^{-\frac{1}{2}}.$$

A Maass form $\phi \in C(SL(2,\mathbb{Z})\backslash\mathfrak{h}^2)$ is called *even* if it satisfies $\phi(-\bar{z}) = \phi(z)$, and is called *odd* if it satisfies $\phi(-\bar{z}) = -\phi(z)$.

The space $\mathcal{E}\left(SL(2,\mathbb{Z})\backslash\mathfrak{h}^2\right)$ is spanned by Eisenstein series $\left\{ E(z, \frac{1}{2} + it) \;\middle|\; t \in \mathbb{R} \right\}$.

In the following we will write $E(z, s)$ as $E_s(z)$.

We will also use μ ($\mathrm{Re}(\mu) = \frac{1}{2}$) instead of s as the parameter of the family of Eisenstein series, as the letter s will be reserved for the complex variable for Rankin–Selberg L-functions.

E_μ has Fourier expansion of the form

$$E_\mu(z) \quad = \quad y^\mu + c(\mu)y^{1-\mu} + \frac{1}{\xi(2\mu)} \sum_{n\neq 0} \sigma_{1-2\mu}(n)\,|n|^{-\frac{1}{2}} \cdot W_\mu(nz) \tag{1.3.2}$$

where $\xi(\mu) = \pi^{-\mu/2}\Gamma(\mu/2)\zeta(\mu)$ is the complete Riemann zeta function, $c(\mu) := \frac{\xi(2\mu-1)}{\xi(\mu)}$,

$$\sigma_\mu(n) = \sum_{d|n} d^\mu$$

is the divisor function and

$$W_\mu(z) := 2|y|^{\frac{1}{2}} K_{\mu-\frac{1}{2}}(2\pi|y|)e(nx) \tag{1.3.3}$$

is the Whittaker function. (See Theorem 3.1.8 in [12].)

We will also write $\lambda_t^{\mathrm{Eis}}(n) = \sigma_{1-2\mu}(n)$ to denote the n-th Hecke eigenvalue of $E_{\frac{1}{2}+it}$, and write $\rho_t^{\mathrm{Eis}}(n) = \xi(2\mu)^{-1}\sigma_{1-2\mu}(n)\,|n|^{-\frac{1}{2}}$ to denote the n-th Fourier coefficient of $E_{\frac{1}{2}+it}$ for simplicity.

1.4 Outline

In Chapter 2, we prepare the theory of the Rankin–Selberg convolution for a $GL(3)$ Eisenstein series (minimal parabolic or maximal parabolic twisted by a $GL(2)$ Maass form) with another $GL(2)$ form, either an Eisenstein series or a Maass form.

In Chapter 3, we give the proof of Theorem 1.2.1. We first prepare all the lemmas needed in the proof, especially the Voronoi formula and the approximate functional equation for $GL(3)$ Eisenstein series. Then we bring out the estimation in details and obtain the desired bound. The proof of Theorem 1.2.4 is similar and we omit the details.

Chapter 2

Rankin–Selberg convolution of a $GL(3)$ Eisenstein series with a $GL(2)$ form

We shall define the Rankin–Selberg convolution of a $GL(3)$ Eisenstein series (minimal parabolic or maximal parabolic, respectively) with a $GL(2)$ form (a Hecke-Maass form or an Eisenstein series, respectively) by Dirichlet series constructed from the Fourier coefficients of the two forms (where the constant terms are discarded).

2.1 $GL(3)$ minimal parabolic Eisenstein series

We shall use the same notations as in [12]. Let the generalized upper half plane $\mathfrak{h}^3$ associated to $GL(3,\mathbb{R})$ be the set of all 3×3 matrices of the form $z = x\cdot y$ where

$$x = \begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}, \tag{2.1.1}$$

with $x_i \in \mathbb{R}$ for $1 \le i \le 3$ and $y_i > 0$. By [12], $\mathfrak{h}^3 \cong GL(3,\mathbb{R})/(O(3,\mathbb{R})\cdot\mathbb{R}^\times$.

Let

$$P_{\min} := \left\{ \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \right\} \cap SL(3,\mathbb{Z}).$$

Let $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$. For $z \in \mathfrak{h}^3$, define

$$I_\nu(z) := y_1^{\nu_1+2\nu_2} y_2^{2\nu_1+\nu_2}.$$

We define the $GL(3)$ minimal parabolic Eisenstein series $\mathscr{E}_{\min,\nu}$ by

$$\mathscr{E}_{\min,\nu}(z) := \sum_{\gamma \in P_{\min}\backslash SL_3(\mathbb{Z})} I_\nu(\gamma z). \tag{2.1.2}$$

$\mathscr{E}_{\min,\nu}$ is well-defined, converges absolutely and uniformly on compact subsets of $\mathfrak{h}^3$ to a $SL(3,\mathbb{Z})$ invariant function provided $\mathrm{Re}(\nu_1)$ and $\mathrm{Re}(\nu_2)$ sufficiently large. (c.f. [12])

It is well-known that $\mathscr{E}_{\min,\nu}$ has Fourier expansion of the form

$$\mathscr{E}_{\min,\nu}(z) \quad = \quad C(z,\nu) + \sum_{\gamma \in U_2(\mathbb{Z})\backslash\Gamma} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_\nu(m,n) \cdot W_{\text{Jacquet}}\left(\begin{pmatrix} |m_1 m_2| & & \\ & |m_1| & \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, \nu, \psi_{1,\frac{m_2}{|m_2|}}\right). \tag{2.1.3}$$

Here $C(z,\nu)$ denotes the degenerate terms in the Fourier expansion associated to $m_1 = 0$ or $m_2 = 0$, and for a character ψ of $U_3(\mathbb{R})$,

$$W_{\text{Jacquet}}(z,u,\psi) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_u(w_0 u z)\overline{\psi(u)}\, dv_1 dv_2 dv_3$$

and $w_0 = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}$, $u = \begin{pmatrix} 1 & v_2 & v_3 \\ & 1 & v_1 \\ & & 1 \end{pmatrix}$. See Theorem 10.8.1 in [12].

In the case of $\nu_0 = \left(\frac{1}{3}, \frac{1}{3}\right)$, we have

$$A_{\nu_0}(m,1) \quad = \quad \sum_{c_1c_2c_3=m} 1 \;=\; d_3(m) \;=\; O_\varepsilon(m^\varepsilon),, \tag{2.1.4}$$

$$A_{\nu_0}(1,m) \quad = \quad \overline{A_{\nu_0}(m,1)} \;=\; d_3(m), \tag{2.1.5}$$

$$\begin{aligned} A_{\nu_0}(m,n) \quad &= \quad \sum_{d\,|\,(m,n)} \mu(d)\, A_{\nu_0}\left(\frac{m}{d},1\right) A_{\nu_0}\left(1,\frac{n}{d}\right) \\ &= \quad \sum_{d\,|\,(m,n)} \mu(d)\, d_3\left(\frac{m}{d}\right) d_3\left(\frac{n}{d}\right). \end{aligned} \tag{2.1.6}$$

where μ denotes the Möbius μ-function

$$\mu(n) = \begin{cases} 1 & \text{if } n \text{ is a square-free positive integer with an even number of prime factors,} \\ -1 & \text{if } n \text{ is a square-free positive integer with an odd number of prime factors,} \\ 0 & \text{if } n \text{ is not square free.} \end{cases}$$

2.2 $GL(3)$ maximal parabolic Eisenstein series

Let

$$P_{2,1} := \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\} \bigcap SL(3,\mathbb{Z}).$$

For $z \in \mathfrak{h}^3$, we define

$$\mathfrak{m}_{P_{2,1}}(z) := \begin{pmatrix} 1 & x_{1,2} & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let ψ be a $GL(2)$ Hecke-Maass form of type w' whose Fourier expansion is

$$\psi(z) = \sum_{n \neq 0} b_n n^{-\frac{1}{2}} W_{w'}(nz). \tag{2.2.1}$$

For $z \in \mathfrak{h}^3$, $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda)$ sufficiently large, we define the $GL(3)$ maximal parabolic Eisenstein series $\mathscr{E}_{\max,\psi,\lambda}$ twisted by the $GL(2)$ Maass form ψ by

$$\mathscr{E}_{\max,\psi,\lambda}(z) := \sum_{\gamma \in P_{2,1} \backslash SL(3,\mathbb{Z})} \mathrm{Det}(\gamma z)^{\lambda} \cdot \psi(\mathfrak{m}_{P_{2,1}}(\gamma z)) \tag{2.2.2}$$

where $\mathrm{Det}(z)^{\lambda} := \left(y_1^2 y_2\right)^{\lambda}$. (c.f. §10.5 in [12])

Then according to Proposition 10.9.3 in [12], the $(n,1)$-th Fourier coefficient of $\mathscr{E}_{\max,\psi,\lambda}$ is

$$A_{\psi,\lambda}(n,1) = n^{-1} \sum_{k_1 k_2 = n} b_{k_1} k_1^{\frac{1}{2}+\lambda} k_2^{-2\lambda+2} = \sum_{k|n} b_k n^{\lambda+1} k^{3\lambda-\frac{3}{2}},$$

where b_k is the k-th Fourier coefficient of $\psi(z)$, as in (2.2.1).

2.3 L-functions associated to $\mathscr{E}_{\mathbf{min},\nu}$, $\mathscr{E}_{\mathbf{max},\psi,\lambda}$, E_μ and ϕ

Let $\mathscr{E}_{\min,\nu}$ and $\mathscr{E}_{\max,\psi,\lambda}$ be the $GL(3)$ Eisenstein series defined in (2.1.2) and (2.2.2) with Fourier expansions (2.1.3) and (2.2.3), respectively.

Let E_μ be a $GL(2)$ Eisenstein series defined in (**??**) with Fourier expansion

$$E_\mu(z) \quad = \quad y^\mu + c(\mu)y^{1-\mu} + \frac{1}{\xi(2\mu)} \sum_{n\neq 0} \eta(\mu, n)\,|n|^{-\frac{1}{2}} \cdot W_\mu(nz) \tag{2.3.1}$$

and ϕ be an even $GL(2)$ Hecke–Maass form of type w with Fourier expansion

$$\phi(z) = \sum_{n\neq 0} a_n n^{-\frac{1}{2}} W_w(nz).$$

Then we define the L-functions associated to $\mathscr{E}_{\min,\nu}$ and $\mathscr{E}_{\max,\psi,\lambda}$, E_μ and ϕ as follows:

$$L(s, \mathscr{E}_{\min,\nu}) \quad := \quad \sum_{n=1}^{\infty} A_\nu(n,1) \cdot n^{-s}, \tag{2.3.2}$$

$$L(s, \mathscr{E}_{\max,\psi,\lambda}) \quad := \quad \sum_{n=1}^{\infty} A_{\psi,\lambda}(n,1) \cdot n^{-s}, \tag{2.3.3}$$

$$L(s, E_\mu) \quad := \quad \sum_{n=1}^{\infty} \sigma_{1-2\mu}(n) \cdot n^{-s}, \tag{2.3.4}$$

$$L(s, \phi) \quad := \quad \sum_{n=1}^{\infty} a_n n^{-s}. \tag{2.3.5}$$

They have Euler products due to the Hecke theory, and as shown in Sections 3.13, 3.14 and 10.8 in [12],

$$\begin{aligned}
L(s, \mathscr{E}_{\min,\nu}) &= \zeta(s + \nu_1 + 2\nu_2 - 1)\zeta(s - 2\nu_1 - \nu_2 + 1)\zeta(s + \nu_1 - \nu_2), \\
L(s, \mathscr{E}_{\max,\psi,\lambda}) &= \zeta(s - \nu_2 - 1)L\left(s - \nu_1 + \frac{1}{2}, \psi\right), \\
L(s, E_\mu) &= \zeta(s + \nu - \frac{1}{2})\zeta(s - \nu + \frac{1}{2}), \\
L(s, \phi) &= \prod_p (1 - \alpha_p p^{-s})^{-1} \prod_p (1 - \beta_p p^{-s})^{-1}.
\end{aligned}$$

Here we only prove the second equality as an example:

$$
\begin{aligned}
L(s, \mathscr{E}_{\max,\psi,\lambda}) &= \sum_{n=1}^{\infty} \sum_{k|n} \frac{b_k\, k^{3\lambda-\frac{3}{2}}}{n^{s+2\lambda-1}} \\
&\overset{(n'=\frac{n}{k})}{=} \sum_{n'=1}^{\infty} \sum_{k} \frac{1}{(n')^{s+2\lambda-1}} \cdot \frac{b_k}{k^{s+2\lambda-1-(3\lambda-\frac{3}{2})}} \\
&= \zeta(s+2\lambda-1) L\left(s-\lambda+\frac{1}{2}, \psi\right).
\end{aligned}
$$

Now we choose $\nu = \nu_0 = (\frac{1}{3}, \frac{1}{3})$ so that $L(s, \mathscr{E}_{\min,\nu_0})$ takes the following simple form:

$$
L(s, \mathscr{E}_{\min,\nu_0}) = \zeta^3(s) = \prod_p \prod_{i=1}^{3} \left(1 - \frac{1}{p^s}\right)^{-1}.
$$

Similarly, we choose $\lambda = \frac{1}{2}$ so that $L(s, \mathscr{E}_{\max,\psi,\lambda})$ takes the following simple form:

$$
L(s, \mathscr{E}_{\max,\psi,\frac{1}{2}}) = \zeta(s) L(s,\psi) = \prod_p \left(1 - p^{-s}\right)^{-1} \left(1 - \alpha'_p p^{-s}\right)^{-1} \left(1 - \beta'_p p^{-s}\right).
$$

2.4 Rankin–Selberg convolutions

We shall study the following Rankin–Selberg convolutions

$$
L(s, \mathscr{E}_{\min,\nu} \times E_\mu) := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{A_\nu(m,n)\overline{\sigma_{1-2\nu}(n)}}{(m^2 n)^s}, \tag{2.4.1}
$$

$$
L(s, \mathscr{E}_{\min,\nu} \times \phi) := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{A_\nu(m,n) a_n}{(m^2 n)^s}. \tag{2.4.2}
$$

$$
L(s, \mathscr{E}_{\max,\psi,\lambda} \times E_\mu) := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{A_{\psi,u}(m,n)\overline{\eta(n,\nu)}}{(m^2 n)^s} \tag{2.4.3}
$$

$$
L(s, \mathscr{E}_{\max,\psi,\lambda} \times \phi) := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{A_{\psi,u}(m,n) a_n}{(m^2 n)^s}. \tag{2.4.4}
$$

They are convergent for $\mathrm{Re}(s)$ sufficiently large. See Section 12.2 in [12].

Theorem 12.3.5 in [12] implies that if f and g are $SL(3,\mathbb{Z})$ and $SL(2,\mathbb{Z})$ forms, respectively, with Euler products

$$L_f(s) = \sum_{n=1}^{\infty} \frac{A(n,1)}{n^s} = \prod_p \prod_{i=1}^{3} (1-\alpha_{p,i} p^{-s})^{-1},$$
$$L_g(s) = \sum_{n=1}^{\infty} \frac{B(n)}{n^s} = \prod_p \prod_{i=1}^{2} (1-\beta_{p,i} p^{-s})^{-1},$$

then

$$L_{f\times g}(s) := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{A(n,m)B(n)}{(m^2 n)^s}$$

has the Euler product

$$L_{f\times g}(s) = \prod_p \prod_{i=1}^{3} \prod_{j=1}^{2} (1-\alpha_{p,i}\overline{\beta_{p,j}} p^{-s})^{-1}.$$

Now we apply this theorem to (2.4.1) and (2.4.2), and take the special choice $\nu_0 = \left(\frac{1}{3}, \frac{1}{3}\right)$. We have

$$L(s, \mathscr{E}_{\min,\nu_0} \times E_\mu) = \prod_p \prod_{i=1}^{3} \left(1 - \frac{1 \cdot p^{\frac{1}{2}-\nu}}{p^s}\right)^{-1} \left(1 - \frac{1 \cdot p^{-\frac{1}{2}+\nu}}{p^s}\right)^{-1} = [L(s, E_\mu)]^3, \tag{2.4.5}$$
$$L(s, \mathscr{E}_{\min,\nu_0} \times \phi) = \prod_p \prod_{i=1}^{3} \left(1 - \frac{\alpha}{p^s}\right)^{-1} \left(1 - \frac{\beta}{p^s}\right)^{-1} = [L(s, \phi)]^3. \tag{2.4.6}$$

If we further take $\nu = \frac{1}{2} + it$ ($t \in \mathbb{R}$), then

$$L(s, E_{\frac{1}{2}+it}) = \zeta(s+it)\zeta(s-it), \tag{2.4.7}$$

and hence

$$L(s, \mathscr{E}_{\min,\nu_0} \times E_{\frac{1}{2}+it}) = \zeta^3(s+it)\zeta^3(s-it). \tag{2.4.8}$$

Therefore,

$$L\left(\frac{1}{2}, \mathscr{E}_{\min,\nu_0} \times E_{\frac{1}{2}+it}\right) = \zeta^3\left(\frac{1}{2}+it\right)\zeta^3\left(\frac{1}{2}-it\right) = \left|\zeta\left(\frac{1}{2}+it\right)\right|^6. \tag{2.4.9}$$

This explains how $\left|\zeta\left(\frac{1}{2}+it\right)\right|^6$ and $[L(s,\phi)]^3$ enter the main theorem naturally.

Similarly, we have

$$\begin{aligned}
L(s,\mathscr{E}_{\max,\psi,\lambda}\times E_\mu) &= \prod_p \left(1-\frac{p^{-2\lambda+1}\cdot p^{\frac{1}{2}-\mu}}{p^s}\right)^{-1}\left(1-\frac{p^{-2\lambda+1}\cdot p^{-\frac{1}{2}+\mu}}{p^s}\right)^{-1} \\
&\quad\cdot\left(1-\frac{\alpha'_p\cdot p^{\lambda-\frac{1}{2}}\cdot p^{\frac{1}{2}-\mu}}{p^s}\right)^{-1}\left(1-\frac{\alpha'_p\cdot p^{\lambda-\frac{1}{2}}\cdot p^{\frac{1}{2}-\mu}}{p^s}\right)^{-1} \\
&\quad\cdot\left(1-\frac{\beta'_p\cdot p^{\lambda-\frac{1}{2}}\cdot p^{-\frac{1}{2}+\mu}}{p^s}\right)^{-1}\left(1-\frac{\beta'_p\cdot p^{\lambda-\frac{1}{2}}\cdot p^{-\frac{1}{2}+\mu}}{p^s}\right)^{-1} \\
&= \zeta\left(s+2\lambda+\mu-\frac{3}{2}\right)\zeta\left(s+2\lambda-\mu-\frac{1}{2}\right)L(s-\lambda+\mu,\psi)L(s-\lambda-\mu+1),
\end{aligned} \tag{2.4.10}$$

$$\begin{aligned}
L(s,\mathscr{E}_{\max,\psi,\lambda}\times \phi) &= \prod_p \left(1-\frac{p^{-2\lambda+1}\cdot\alpha_p\cdot p^{\frac{1}{2}-\mu}}{p^s}\right)^{-1}\left(1-\frac{p^{-2\lambda+1}\cdot\beta_p\cdot p^{-\frac{1}{2}+\mu}}{p^s}\right)^{-1} \\
&\quad\cdot\left(1-\frac{\alpha'_p\cdot p^{\lambda-\frac{1}{2}}\cdot\alpha_p\cdot p^{\frac{1}{2}-\mu}}{p^s}\right)^{-1}\left(1-\frac{\alpha'_p\cdot p^{\lambda-\frac{1}{2}}\cdot\beta_p\cdot p^{\frac{1}{2}-\mu}}{p^s}\right)^{-1} \\
&\quad\cdot\left(1-\frac{\beta'_p\cdot p^{\lambda-\frac{1}{2}}\cdot\alpha_p\cdot p^{-\frac{1}{2}+\mu}}{p^s}\right)^{-1}\left(1-\frac{\beta'_p\cdot p^{\lambda-\frac{1}{2}}\cdot\beta_p\cdot p^{-\frac{1}{2}+\mu}}{p^s}\right)^{-1} \\
&= L(s+2\lambda-1,\phi)L(s-\lambda+\frac{1}{2},\psi\times\phi).
\end{aligned} \tag{2.4.11}$$

Choose $\lambda=\frac{1}{2}$ and $\mu=\frac{1}{2}+it$, we have

$$\begin{aligned}
L(s,\mathscr{E}_{\max,\psi,\frac{1}{2}}\times E_{\frac{1}{2}+it}) &= \zeta\,(s+it)\,\zeta\,(s-it)\,L(s+it,\psi)L(s-it,\psi), \\
&= \left|\zeta(s+it)L(s+it,\psi)\right|^2,
\end{aligned} \tag{2.4.12}$$

$$L(s,\mathscr{E}_{\max,\psi,\frac{1}{2}}\times\phi) = L(s,\phi)L(s,\psi\times\phi). \tag{2.4.13}$$

The analytic properties of the Rankin–Selberg L-function on the left sides, such as meromorphic continuation and functional equations follow from the right sides. Moreover, we also obtain the nonnegativity of these Rankin–Selberg L-functions at $s=\frac{1}{2}$ in the same way.

For an even Hecke-Maass form ϕ with Laplacian eigenvalue $\frac{1}{4}+t^2$ ($t\geq 0$), we define

$$\Lambda(s,\mathscr{E}_{\min,\nu_0}\times\phi) = \pi^{-3s}\Gamma^3\left(\frac{s-it}{2}\right)\Gamma^3\left(\frac{s+it}{2}\right)L(s,\mathscr{E}_{\min,\nu_0}\times\phi). \tag{2.4.14}$$

By (2.4.6) and Proposition 3.13.5 in [12], it is easy to see that $\Lambda(s,\mathscr{E}_{\min,\nu_0}\times\phi)$ is an entire function and satisfies the following functional equation:

$$\Lambda(s, \mathscr{E}_{\min,\nu_0} \times \phi) = \Lambda(1-s, \mathscr{E}_{\min,\nu_0} \times \phi).$$

(Remark: This is only true for even Maass forms. If ϕ were an odd Maass form, there should be a (-1) factor on the right side of the functional equation.)

For a $GL(2)$ Eisenstein series $E_{\frac{1}{2}+it}$, define

$$\Lambda(s, \mathscr{E}_{\min,\nu_0} \times E_{\frac{1}{2}+it}) \quad := \quad \pi^{-3s}\Gamma^3\left(\frac{s-it}{2}\right)\Gamma^3\left(\frac{s+it}{2}\right)L(s, \mathscr{E}_{\min,\nu_0} \times E_{\frac{1}{2}+it}). \tag{2.4.15}$$

By (2.4.9), we see that $\Lambda(s, \mathscr{E}_{\min,\nu_0} \times E_{\frac{1}{2}+it})$ is entire and has functional equation

$$\Lambda(s, \mathscr{E}_{\min,\nu_0} \times E_{\frac{1}{2}+it}) = \Lambda(1-s, \mathscr{E}_{\min,\nu_0} \times E_{\frac{1}{2}+it}).$$

Chapter 3

Proof of the Main Theorems

3.1 Outline of the proof

The left side of (1.2.1) in Theorem 1.2.1 is "a spectral sum" over $SL(2,\mathbb{Z})$ forms. This suggests using a kind of trace formula to transform the spectral sum into a kind of "geometric" sum. In this case, we will use the Kuznetsov trace formula for $SL(2,\mathbb{Z})$ (Proposition 3.4.1).

The "geometric side" thus obtained would be a weighted sum of the product of the Fourier coefficients $A(n,m)$ of the $SL(3,\mathbb{Z})$ Eisenstein series and Kloosterman sums. We will estimate the sum term by term. Various analytic tools are utilized, especially the Voronoi formula.

3.2 Preliminaries

In this section, we establish some lemmas needed for the proof of the main theorems and the corollaries.

Lemma 3.2.1. (Nonegativity of the Rankin–Selberg L-functions)
Let ψ be a $GL(2)$ Maass–Hecke form. Let $\mathscr{E}_{min,\nu_0}$ and $\mathscr{E}_{max,\psi,\frac{1}{2}}$ be the $GL(3)$ Eisenstein series defined as before (see (2.1.2) and (2.2.2)), where $\nu_0 = \left(\frac{1}{3}, \frac{1}{3}\right)$.

(i) Let ϕ be a $GL(2)$ even Maass form. Then

$$L\left(\frac{1}{2}, \mathscr{E}_{min,\nu_0} \times \phi\right) \geq 0, \qquad L\left(\frac{1}{2}, \mathscr{E}_{max,\psi,\frac{1}{2}} \times \phi\right) \geq 0.$$

(ii) Let $E_{\frac{1}{2}+it}$ be the $GL(2)$ Eisenstein series defined as before. Then

$$L\left(\frac{1}{2}, \mathscr{E}_{min,\nu_0} \times E_{\frac{1}{2}+it}\right) \geq 0, \qquad L\left(\frac{1}{2}, \mathscr{E}_{max,\psi,\frac{1}{2}} \times E_{\frac{1}{2}+it}\right) \geq 0.$$

Proof. (i) The nonnegativity of $L\left(\frac{1}{2}, \mathscr{E}_{\min,\nu_0} \times \phi\right)$ follows from (2.4.6) together with the facts $L\left(\frac{1}{2}, \phi\right) \geq 0$ and $L\left(\frac{1}{2}, \psi \times \phi\right) \geq 0$ by [17, 29, 30].

(ii) The nonnegativity of $L\left(\frac{1}{2}, \mathscr{E}_{\min,\nu_0} \times E_{\frac{1}{2}+it}\right)$ and $L\left(\frac{1}{2}, \mathscr{E}_{\max,\psi,\frac{1}{2}} \times \phi\right)$ follows from (2.4.5) and (2.4.9) directly.

□

Lemma 3.2.2. *Let $\mathscr{E}_{min,\nu}$ be the $GL(3)$ Eisenstein series defined in Section 1.3. Let $A_\nu(m,n)$ be the (m,n)-th Fourier coefficients of $\mathscr{E}_{min,\nu}$ (see 2.1.3). Then we have*

$$\sum_{m^2 n \leq N} \sum |A_\nu(m,n)|^2 \ll N. \tag{3.2.1}$$

Proof. See [34, 39, 42]. □

Corollary 3.2.3. *Let $A_\nu(m,n)$ be the Fourier coefficients of $\mathscr{E}_{min,\nu}$ (see 2.1.3). Then we have*

$$\sum_{n \leq N} |A_\nu(m,n)| \ll N|m|. \tag{3.2.2}$$

Proof. Applying Cauchy's inequality to Lemma 3.2.2 and the result follows. □

3.3 Approximate functional equations

We first quote Theorem 5.3 in [25] which is a general result about approximations to L-functions in the critical strip, and then apply it to the Rankin–Selberg L-functions, resulting in Theorem 3.3.2 below.

Theorem 3.3.1. *Let $L(f,s)$ be an L-function (as defined in [25]) with root number $\varepsilon(f)$, conductor $q(f)$ and gamma factor $\gamma(f,s)$, and*

$$\varepsilon(f,s) = \varepsilon(f) q(f)^{\frac{1}{2}-s} \frac{\gamma(f, 1-s)}{\gamma(f,s)}. \tag{3.3.1}$$

Let $\Lambda(f,s) = q(f)^{\frac{s}{2}}\gamma(f,s)L(f,s)$ be the completed L-function for $L(f,s)$.

Let $F(u)$ be any function which is holomorphic and bounded in the strip $-4 < \mathrm{Re}(u) < 4$, even, and normalized by $F(0) = 1$. Let $V_s(y)$ be a smooth function defined by

$$V_s(y) = \frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty} y^{-u}F(u)\frac{\gamma(f,s+u)}{\gamma(f,s)}\frac{du}{u}. \tag{3.3.2}$$

Then for $X > 0$ and s in the strip $0 \le \sigma \le 1$ we have

$$L(f,s) = \sum_n \frac{\lambda_f(n)}{n^s}V_s\left(\frac{n}{X\sqrt{q}}\right) + \varepsilon(f,s)\sum_n \frac{\overline{\lambda_f}(n)}{n^{1-s}}V_{1-s}\left(\frac{nX}{\sqrt{q}}\right) + R. \tag{3.3.3}$$

The last term $R = 0$ if $\Lambda(f,s)$ is entire, otherwise

$$R = (\mathrm{Res}_{u=1-s} + \mathrm{Res}_{u=-s})\,\frac{\Lambda(f,s+u)}{q^{s/2}\gamma(f,s)}\frac{F(u)}{u}X^u. \tag{3.3.4}$$

Proof. See Section 5.2 in [25]. □

The theorem above only applies to $L(f,s)$ which has only one pole at $s = 1$. For our purpose of studying the Rankin–Selberg L-functions (2.4.1–2.4.4), which may have more than one pole, a slight modification is needed. Following the same idea of the proof of Theorem 3.3.1, we see that in the general case the residue term should be changed to

$$R = \sum_i (\mathrm{Res}_{u=z_i-s} + \mathrm{Res}_{u=1+z_i-s})\,\frac{\Lambda(f,s+u)}{q^{s/2}\gamma(f,s)}\frac{F(u)}{u}X^u. \tag{3.3.5}$$

where z_i run over the poles of $L(f,s)$.

Now we apply the modified version of Theorem 3.3.1 to

$$L(s, \mathscr{E}_{\min,\nu_0} \times \phi) \quad = \quad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{A_{\nu_0}(m,n)\lambda_n}{(m^2n)^s} \tag{3.3.6}$$

and take $s = \frac{1}{2}$, $X = 1$. We get

$$L\left(\frac{1}{2}, \mathscr{E}_{\min,\nu_0} \times \phi\right) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{A_{\nu_0}(m,n)\lambda_n}{(m^2n)^{\frac{1}{2}}} V_\phi\left(m^2n\right) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\overline{A_{\nu_0}(m,n)\lambda_n}}{(m^2n)^{\frac{1}{2}}} V_\phi\left(m^2n\right) + R_\phi\left(\frac{1}{2}, t\right) \tag{3.3.7}$$

By (2.4.6), (2.4.14) and (3.3.5), we know $\Lambda(s, \mathscr{E}_{\min,\nu_0} \times \phi)$ is entire and hence

$$R_\phi(s,t) = 0. \tag{3.3.8}$$

Similarly, for

$$L(s, \mathscr{E}_{\min,\nu_0} \times E_{\frac{1}{2}+it}) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{A_{\nu_0}(m,n)\,\overline{\lambda_t^{\mathrm{Eis}}(n)}}{(m^2n)^s},$$

we obtain (in the case $q = 1$ and $\varepsilon(f) = 1$)

$$\begin{aligned} L(\frac{1}{2}, \mathscr{E}_{\min,\nu_0} \times E_{\frac{1}{2}+it}) = & \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{A_{\nu_0}(m,n)\,\overline{\lambda_t^{\mathrm{Eis}}(n)}}{(m^2n)^{\frac{1}{2}}} V\left(m^2n, t\right) \\ & + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\overline{A_{\nu_0}(m,n)}\,\lambda_t^{\mathrm{Eis}}(n)}{(m^2n)^{\frac{1}{2}}} V\left(m^2n, t\right) \\ & + R_{\mathrm{Eis}}\left(\frac{1}{2}, t\right), \end{aligned} \tag{3.3.9}$$

where

$$\begin{aligned} R_{\mathrm{Eis}}(s,t) &= (\mathrm{Res}_{u=1-s+it} + \mathrm{Res}_{u=-s+it} + \mathrm{Res}_{u=1-s-it} + \mathrm{Res}_{u=s-it}) \\ &\quad \cdot \frac{\Lambda(s+u, \mathscr{E}_{\min,\nu_0} \times E_{\frac{1}{2}+it})}{\pi^{-3s}\Gamma^3\left(\frac{s-it}{2}\right)\Gamma^3\left(\frac{s+it}{2}\right)} \frac{F(u)}{u} \\ &= \pi^{3s}\Gamma^{-3}\left(\frac{s-it}{2}\right)\Gamma^{-3}\left(\frac{s+it}{2}\right) \cdot \\ &\quad \cdot (\mathrm{Res}_{u=1-s+it} + \mathrm{Res}_{u=-s+it} + \mathrm{Res}_{u=1-s-it} + \mathrm{Res}_{u=s-it}) \\ &\quad \cdot \left[\xi(s+u-it)\xi(s+u+it)\right]^3 \frac{F(u)}{u}. \end{aligned} \tag{3.3.10}$$

Using the computational software program *Mathematica* to compute $R_{\mathrm{Eis}}\left(\frac{1}{2}, t\right)$, it can be shown that R_{Eis} is relatively small (due to the asymptotic behavior of $\Gamma(s/2)\zeta(s)$ for $s = 1 + it$ as $t \to \pm\infty$) and can be omitted. The residues are listed explicitly as follows:

$$
\begin{aligned}
&\mathrm{Res}_{u=\frac{1}{2}+it} \quad \xi(\frac{1}{2}+u-it)^3\xi(\frac{1}{2}+u+it)^3 \\
= \ & \tfrac{3}{16\pi^{\frac{3}{2}+3it}} \Bigg(\quad 16\gamma^2\Gamma\left(\frac{1}{2}+it\right)^3\zeta(1+2it)^3 + \pi^2\Gamma\left(\frac{1}{2}+it\right)^3\zeta(1+2it)^3 \\
&-48\gamma\Gamma\left(\frac{1}{2}+it\right)^3\log(\pi)\zeta(1+2it)^3 + 24\Gamma\left(\frac{1}{2}+it\right)^3\log(\pi)^2\zeta(1+2it)^3 \\
&+24\gamma\Gamma\left(\frac{1}{2}+it\right)^3\psi_0\left(\frac{1}{2}\right)\zeta(1+2it)^3 - 24\Gamma\left(\frac{1}{2}+it\right)^3\log(\pi)\psi_0\left(\frac{1}{2}\right)\zeta(1+2it)^3 \\
&+6\Gamma\left(\frac{1}{2}+it\right)^3\psi_0\left(\frac{1}{2}\right)^2\zeta(1+2it)^3 + 24\gamma\Gamma\left(\frac{1}{2}+it\right)^3\psi_0\left(\frac{1}{2}+it\right)\zeta(1+2it)^3 \\
&-24\Gamma\left(\frac{1}{2}+it\right)^3\log(\pi)\psi_0\left(\frac{1}{2}+it\right)\zeta(1+2it)^3 \\
&+12\Gamma\left(\frac{1}{2}+it\right)^3\psi_0\left(\frac{1}{2}\right)\psi_0\left(\frac{1}{2}+it\right)\zeta(1+2it)^3 \\
&+6\Gamma\left(\frac{1}{2}+it\right)^3\psi_0\left(\frac{1}{2}+it\right)^2\zeta(1+2it)^3 \\
&+2\Gamma\left(\frac{1}{2}+it\right)^3\psi_1\left(\frac{1}{2}+it\right)\zeta(1+2it)^3 \\
&-16\Gamma\left(\frac{1}{2}+it\right)^3\gamma_1\zeta(1+2it)^3 \\
&+48\gamma\Gamma\left(\frac{1}{2}+it\right)^3\zeta(1+2it)^2\zeta'(1+2it) \\
&-48\Gamma\left(\frac{1}{2}+it\right)^3\log(\pi)\zeta(1+2it)^2\zeta'(1+2it) \\
&+24\Gamma\left(\frac{1}{2}+it\right)^3\psi_0\left(\frac{1}{2}\right)\zeta(1+2it)^2\zeta'(1+2it) \\
&+24\Gamma\left(\frac{1}{2}+it\right)^3\psi_0\left(\frac{1}{2}+it\right)\zeta(1+2it)^2\zeta'(1+2it) \\
&+16\Gamma\left(\frac{1}{2}+it\right)^3\zeta(1+2it)\zeta'(1+2it)^2 \\
&+8\Gamma\left(\frac{1}{2}+it\right)^3\zeta(1+2it)^2\zeta''(1+2it) \quad \Bigg),
\end{aligned}
\tag{3.3.11}
$$

$$
\begin{aligned}
&\mathrm{Res}_{u=\frac{1}{2}-it} \quad \xi(\frac{1}{2}+u-it)^3\xi(\frac{1}{2}+u+it)^3 \\
= \quad & \tfrac{3}{16\pi^2}\Bigg(\quad 16\gamma^2\pi^{\frac{1}{2}+3it}\Gamma\left(\frac{1}{2}-it\right)^3\zeta(1-2it)^3+\pi^{\frac{5}{2}+3it}\Gamma\left(\frac{1}{2}-it\right)^3\zeta(1-2it)^3 \\
& -48\gamma\pi^{\frac{1}{2}+3it}\Gamma\left(\frac{1}{2}-it\right)^3\log(\pi)\zeta(1-2it)^3+24\pi^{\frac{1}{2}+3it}\Gamma\left(\frac{1}{2}-it\right)^3\log(\pi)^2\zeta(1-2it)^3 \\
& +24\gamma\pi^{\frac{1}{2}+3it}\Gamma\left(\frac{1}{2}-it\right)^3\psi_0\left(\frac{1}{2}\right)\zeta(1-2it)^3-24\pi^{\frac{1}{2}+3it}\Gamma\left(\frac{1}{2}-it\right)^3\log(\pi)\psi_0\left(\frac{1}{2}\right)\zeta(1-2it)^3 \\
& +6\pi^{\frac{1}{2}+3it}\Gamma\left(\frac{1}{2}-it\right)^3\psi_0\left(\frac{1}{2}\right)^2\zeta(1-2it)^3+24\gamma\pi^{\frac{1}{2}+3it}\Gamma\left(\frac{1}{2}-it\right)^3\psi_0\left(\frac{1}{2}-it\right)\zeta(1-2it)^3 \\
& -24\pi^{\frac{1}{2}+3it}\Gamma\left(\frac{1}{2}-it\right)^3\log(\pi)\psi_0\left(\frac{1}{2}-it\right)\zeta(1-2it)^3 \\
& +12\pi^{\frac{1}{2}+3it}\Gamma\left(\frac{1}{2}-it\right)^3\psi_0\left(\frac{1}{2}\right)\psi_0\left(\frac{1}{2}-it\right)\zeta(1-2it)^3 \\
& +6\pi^{\frac{1}{2}+3it}\Gamma\left(\frac{1}{2}-it\right)^3\psi_0\left(\frac{1}{2}-it\right)^2\zeta(1-2it)^3 \\
& +2\pi^{\frac{1}{2}+3it}\Gamma\left(\frac{1}{2}-it\right)^3\psi_1\left(\frac{1}{2}-it\right)\zeta(1-2it)^3 \\
& -16\pi^{\frac{1}{2}+3it}\Gamma\left(\frac{1}{2}-it\right)^3\gamma_1\zeta(1-2it)^3 \\
& +48\gamma\pi^{\frac{1}{2}+3it}\Gamma\left(\frac{1}{2}-it\right)^3\zeta(1-2it)^2\zeta'(1-2it) \\
& -48\pi^{\frac{1}{2}+3it}\Gamma\left(\frac{1}{2}-it\right)^3\log(\pi)\zeta(1-2it)^2\zeta'(1-2it) \\
& +24\pi^{\frac{1}{2}+3it}\Gamma\left(\frac{1}{2}-it\right)^3\psi_0\left(\frac{1}{2}\right)\zeta(1-2it)^2\zeta'(1-2it) \\
& +24\pi^{\frac{1}{2}+3it}\Gamma\left(\frac{1}{2}-it\right)^3\psi_0\left(\frac{1}{2}-it\right)\zeta(1-2it)^2\zeta'(1-2it) \\
& +16\pi^{\frac{1}{2}+3it}\Gamma\left(\frac{1}{2}-it\right)^3\zeta(1-2it)\zeta'(1-2it)^2 \\
& +8\pi^{\frac{1}{2}+3it}\Gamma\left(\frac{1}{2}-it\right)^3\zeta(1-2it)^2\zeta''(1-2it) \quad \Bigg),
\end{aligned} \tag{3.3.12}
$$

$$
\begin{aligned}
&\mathrm{Res}_{u=-\frac{1}{2}+it} \quad \xi(\frac{1}{2}+u-it)^3\xi(\frac{1}{2}+u+it)^3 \\
= \quad & \tfrac{3}{16}\pi^{-3it}\Big(\quad 2\gamma^2\Gamma(it)^3\zeta(2it)^3 - \pi^2\Gamma(it)^3\zeta(2it)^3 \\
& -8\Gamma(it)^3\log(2)^2\zeta(2it)^3 - 24\gamma\Gamma(it)^3\log(\pi)\zeta(2it)^3 \\
& -16\Gamma(it)^3\log(2)\log(\pi)\zeta(2it)^3 - 32\Gamma(it)^3\log(\pi)^2\zeta(2it)^3 + 24\gamma\Gamma(it)^3\log(2\pi)\zeta(2it)^3 \\
& +48\Gamma(it)^3\log(\pi)\log(2\pi)\zeta(2it)^3 - 16\Gamma(it)^3\log(2\pi)^2\zeta(2it)^3 \\
& +12\gamma\Gamma(it)^3\psi_0(it)\zeta(2it)^3 + 24\Gamma(it)^3\log(\pi)\psi_0(it)\zeta(2it)^3 \\
& -24\Gamma(it)^3\log(2\pi)\psi_0(it)\zeta(2it)^3 - 6\Gamma(it)^3\psi_0(it)^2\zeta(2it)^3 \\
& -2\Gamma(it)^3\psi_1(it)\zeta(2it)^3 + 16\Gamma(it)^3\gamma_1\zeta(2it)^3 \\
& +24\gamma\Gamma(it)^3\zeta(2it)^2\zeta'(2it) + 48\Gamma(it)^3\log(\pi)\zeta(2it)^2\zeta'(2it) \\
& -48\Gamma(it)^3\log(2\pi)\zeta(2it)^2\zeta'(2it) - 24\Gamma(it)^3\psi_0(it)\zeta(2it)^2\zeta'(2it) \\
& -16\Gamma(it)^3\zeta(2it)\zeta'(2it)^2 - 8\Gamma(it)^3\zeta(2it)^2\zeta''(2it) \quad \Big),
\end{aligned}
\tag{3.3.13}
$$

$$
\begin{aligned}
&\mathrm{Res}_{u=-\frac{1}{2}-it} \quad \xi(\frac{1}{2}+u-it)^3\xi(\frac{1}{2}+u+it)^3 \\
= \quad & \tfrac{3}{16}\Big(\quad -\pi^{2+3it}\Gamma(-it)^3\zeta(-2it)^3 + 2\gamma^2\pi^{3it}\Gamma(-it)^3\zeta(-2it)^3 \\
& -8\pi^{3it}\Gamma(-it)^3\log(2)^2\zeta(-2it)^3 - 24\gamma\pi^{3it}\Gamma(-it)^3\log(\pi)\zeta(-2it)^3 \\
& -16\pi^{3it}\Gamma(-it)^3\log(2)\log(\pi)\zeta(-2it)^3 - 32\pi^{3it}\Gamma(-it)^3\log(\pi)^2\zeta(-2it)^3 \\
& +24\gamma\pi^{3it}\Gamma(-it)^3\log(2\pi)\zeta(-2it)^3 + 48\pi^{3it}\Gamma(-it)^3\log(\pi)\log(2\pi)\zeta(-2it)^3 \\
& -16\pi^{3it}\Gamma(-it)^3\log(2\pi)^2\zeta(-2it)^3 + 12\gamma\pi^{3it}\Gamma(-it)^3\psi_0(-it)\zeta(-2it)^3 \\
& +24\pi^{3it}\Gamma(-it)^3\log(\pi)\psi_0(-it)\zeta(-2it)^3 - 24\pi^{3it}\Gamma(-it)^3\log(2\pi)\psi_0(-it)\zeta(-2it)^3 \\
& -6\pi^{3it}\Gamma(-it)^3\psi_0(-it)^2\zeta(-2it)^3 - 2\pi^{3it}\Gamma(-it)^3\psi_1(-it)\zeta(-2it)^3 \\
& +16\pi^{3it}\Gamma(-it)^3\gamma_1\zeta(-2it)^3 + 24\gamma\pi^{3it}\Gamma(-it)^3\zeta(-2it)^2\zeta'(-2it) \\
& +48\pi^{3it}\Gamma(-it)^3\log(\pi)\zeta(-2it)^2\zeta'(-2it) - 48\pi^{3it}\Gamma(-it)^3\log(2\pi)\zeta(-2it)^2\zeta'(-2it) \\
& -24\pi^{3it}\Gamma(-it)^3\psi_0(-it)\zeta(-2it)^2\zeta'(-2it) - 16\pi^{3it}\Gamma(-it)^3\zeta(-2it)\zeta'(-2it)^2 \\
& -8\pi^{3it}\Gamma(-it)^3\zeta(-2it)^2\zeta''(-2it) \quad \Big).
\end{aligned}
\tag{3.3.14}
$$

Here

$$\psi_n(z) = \frac{d^{n+1}}{dz^{n+1}} \ln[\Gamma(z)],$$
$$\gamma_n = \lim_{m\to\infty}\left(\sum_{n=1}^{m} \frac{(\ln k)^n}{k} - \frac{(\ln m)^{n+1}}{n+1}\right).$$

where γ_n is known as Stielje's constant.

In summary, we obtain the desired approximate functional equations for $L\left(\frac{1}{2}, \mathscr{E}_{\min,\nu_0} \times \phi\right)$ and $L\left(\frac{1}{2}, \mathscr{E}_{\min,\nu_0} \times E_{\frac{1}{2}+it}\right)$.

Theorem 3.3.2. *Let $\mathscr{E}_{min,\nu_0}$, $E_{\frac{1}{2}+it}$ be the Eisenstein series as defined in (2.1.2) and (**??**). Let $F(u)$ be any function which is holomorphic and bounded in the strip $-4 < \mathrm{Re}(u) < 4$, even, and normalized by $h(0) = 1$. Let $X > 0$. Then for s in the strip $0 \le \sigma \le 1$ we have*

$$L\left(\frac{1}{2}, \mathscr{E}_{min,\nu_0} \times u_j\right) = 2\sum_{m\ge1}\sum_{n\ge1} \frac{A(n,m)\lambda_j(n)}{(m^2n)^{\frac{1}{2}}} V(m^2n, t_j), \tag{3.3.15}$$

$$L\left(\frac{1}{2}, \mathscr{E}_{min,\nu_0} \times E_{\frac{1}{2}+it}\right) = 2\sum_{m\ge1}\sum_{n\ge1} \mathrm{Re}\, \frac{A(n,m)\, \overline{\lambda_t^{Eis}(n)}}{(m^2n)^{\frac{1}{2}}} V(m^2n, t) + R_{Eis}\left(\frac{1}{2}, t\right) \tag{3.3.16}$$

where

$$V(y,t) = \frac{1}{2\pi i}\int_{\mathrm{Re}\, u=3} y^{-u} F(u) \frac{\gamma\left(\frac{1}{2}+u, t\right)}{\gamma\left(\frac{1}{2}, t\right)} \frac{du}{u} \tag{3.3.17}$$

and

$$\gamma(s,t) := \pi^{-3s}\left(\Gamma\left(\frac{s-it}{2}\right)\Gamma\left(\frac{s+it-\gamma}{2}\right)\right)^3. \tag{3.3.18}$$

Here $R_{Eis}\left(\frac{1}{2}, t\right)$ is the sum of (3.3.11), (3.3.12), (3.3.13) and (3.3.14).

The following is Lemma 2.3 in [34] which describes the growth of $V(y,t)$ which appears in Theorem 3.3.1. Since it is needed to effectively bound the terms in (3.3.15) and (3.3.16), we include it here.

Lemma 3.3.3. *Set*

$$F(u) = \left(\cos\frac{\pi u}{A}\right)^{-3A}$$

for $\operatorname{Im} t \leq 1000$*, where A is a positive integer, and*

$$V(y,t) := \frac{1}{2\pi i}\int_{\operatorname{Re} u=1000} y^{-u} F(u) \frac{\gamma\left(\frac{1}{2}+u,t\right)}{\gamma\left(\frac{1}{2},t\right)}\frac{du}{u}. \tag{3.3.19}$$

Here

$$\begin{aligned}\gamma(s,t) \quad := \quad & \pi^{-3s}\Gamma\left(\frac{s-it-\alpha}{2}\right)\Gamma\left(\frac{s-it-\beta}{2}\right)\Gamma\left(\frac{s-it-\gamma}{2}\right)\\ & \Gamma\left(\frac{s+it-\alpha}{2}\right)\Gamma\left(\frac{s+it-\beta}{2}\right)\Gamma\left(\frac{s+it-\gamma}{2}\right)\end{aligned}$$

for some constants α*,* β *and* γ*.*

Then for $y > 0$*,* $t > 0$*,* $i = 1, 2$*,*

(i) the derivatives of $V(y,t)$ *with respect to y satisfy*

$$y^a\frac{\partial^a}{\partial y^a}V(y,t) \ll \left(1+\frac{y}{|t|^3}\right)^{-A}, \tag{3.3.20}$$

$$y^a\frac{\partial^a}{\partial y^a}V(y,t) = \delta_a + O\left(\left(\frac{y}{|t|^3}\right)^c\right), \tag{3.3.21}$$

where $0 < c \leq \frac{1}{6}$*,* $\delta_0 = 1$ *for* $a = 0$ *and* $\delta = 0$ *for* $a \neq 0$*, and the implied constants depend only on c, a, A.*

(ii) If $1 \leq y \leq t^{3+\varepsilon}$*, then as* $t \to \infty$*, we have*

$$\begin{aligned}V(y,t) \quad = \quad & \frac{1}{2\pi i}\int_{\left(\frac{1}{2}\right)}\left(\frac{t^3}{8\pi^3 y}\right)^u F(u)\left[1+\frac{p_1(\operatorname{Im} u)}{t}+\cdots+\frac{p_{n-1}(\operatorname{Im} u)}{t^{n-1}}+O\left(\frac{p_n(\operatorname{Im} u)}{t^n}\right)\right]\frac{du}{u}\\ & +O(t^{-B})\end{aligned} \tag{3.3.22}$$

where p_i *are polynomials and B is arbitrarily large.*

Proof. The same as in [34]. □

3.4 The Kuznetsov trace formula for $GL(2)$

The Kuznetsov trace formula ([27]) is a special form of relative trace formula. It plays a key role here since it transforms the spectral sum into a sum of weighted Kloosterman sums which we have tools to estimate.

We adopt the same notation as in Section 1.3. Let $\left\{u_j\right\}$ be the $GL(2)$ Hecke–Maass eigenforms spanning $C\left(SL(2,\mathbb{Z})\backslash\mathfrak{h}^2\right)$ with Laplacian eigenvalues t_j and Hecke eigenvalues $\lambda_j(n)$. Let $\left\{E_{2,\frac{1}{2}+it}\right\}$ be the $GL(2)$ Eisenstein series spanning $\mathcal{E}\left(SL(2,\mathbb{Z})\backslash\mathfrak{h}^2\right)$ with Hecke eigenvalues $\lambda_t^{\mathrm{Eis}}(n)$.

Next, we take a test function $h(t)$ which is even and is assumed to satisfy the following conditions:

(i) $h(t)$ is holomorphic in $|\mathrm{Im}\, t| \le \frac{1}{2}+\varepsilon$;

(ii) $h(t) \ll (|t|+1)^{-2-\varepsilon}$ in the above strip.

We set

$$\begin{aligned}
\omega_j &= 4\pi|\rho_j(1)|^2/\cosh \pi t_j,\\
\omega(t) &= 4\pi\left|\rho_t^{\mathrm{Eis}}(1)\right|^2\cosh^{-1}\pi t,\\
H &= \frac{2}{\pi}\int_0^\infty h(t)\tanh(\pi t)t dt,\\
H^+(x) &= 2i\int_{-\infty}^{\infty} J_{2it}(x)\frac{h(t)t}{\cosh \pi t}dt,\\
H^-(x) &= \frac{4}{\pi}\int_{-\infty}^{\infty} K_{2it}(x)\sinh(\pi t)h(t)t dt.
\end{aligned} \tag{3.4.1}$$

In the above, $J_\nu(x)$ and $K_\nu(x)$ are the standard J-Bessel function and K-Bessel function respectively.

Proposition 3.4.1. *With the above notations, for any $m,\ n \ge 1$, we have the following Kuznetsov formula*

$$\begin{aligned}
&\sum_{j\ge1}{}' h(t_j)\omega_j\lambda_j(m)\lambda_j(n) + \frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)\omega(t)\overline{\lambda}_t^{Eis}(m)\lambda_t^{Eis}(n)\,dt\\
&= \frac{1}{2}\delta(m,n)H + \sum_{c>0}\frac{1}{2c}\left\{S(m,n;c)H^+\left(\frac{4\pi\sqrt{mn}}{c}\right) + S(-m,n;c)H^-\left(\frac{4\pi\sqrt{mn}}{c}\right)\right\}
\end{aligned} \tag{3.4.2}$$

where $\sum'$ *is restricted to the even Maass forms,* $\delta(m,n)$ *is the Kronecker symbol, and*

$$S(a,b;c) = \sum_{d\bar{d}\equiv 1(mod\ c)} e\Big(\frac{da+\bar{d}b}{c}\Big)$$

is the classical Kloosterman sum.

Proof. See [6]. □

3.5 The Voronoi summation formula for a $GL(3)$ Eisenstein series

The Voronoi summation formulae are generalizations of the Poisson summation formula. The sums are weighted by Fourier coefficients of automorphic forms, possibly with twists. The Voronoi formula for $GL(2)$ has served as a fundamental analytic tool to study the subconvexity problem. For a survey, see [22].

The Voronoi formulae relate sums of the form

$$\sum_{n\in\mathbb{Z}} a_n e(n\alpha) f(n) = \sum_{n\in\mathbb{Z}} a_n S(n,\alpha) F(n)$$

where a_n are Fourier coefficients of an automorphic form, and $\alpha \in \mathbb{Q}$. On the right side, $S(m,\alpha)$ is an exponential sum. (For the $GL(2)$ case, $S(m,\alpha)$ is a single exponential, while for $GL(3)$ it is a Kloosterman sum.) Finally, f and F are a pair of test functions related by an integral transform, an analogue of the Fourier transform in the Poisson summation formula.

The formula is useful for estimating the sum on the left side, since the right side is a sum of integral transforms of functions which decay rapidly (c.f. [40], Section 1).

A classical approach to prove the Voronoi formula for $GL(2)$ is to apply Mellin inversion to the functional equation of the standard L-function with twists. In other words, the starting point of the proof is the modularity with respect to $z \mapsto -1/z$ of the automorphic forms in concern.

The Voronoi formula for $GL(3)$ Maass forms with twists by additive characters was first proven by Miller and Schmid using the theory of automorphic distributions [41]. Goldfeld and Li developed a purely analytic proof in [13] and [14] in the spirit of taking Mellin inversion of functional equation of certain L-functions.

The main result of this section, the Voronoi formula for $GL(3)$ Eisenstein series (Proposition 3.5.2), can be derived in a similar manner, yet one needs to take care of the residue terms coming from the pole of Eisenstein series. In an unpublished notes by X. Li [32], the Voronoi formula for the triple divisor function is established by using a minimal $GL(3)$ Eisenstein series. Here we closely follow her idea with minor modifications.

In the following, we will sketch the proof in [13] for the Voronoi formula for $GL(3)$ Maass forms. Then we modify the proof to obtain the Voronoi formula for $GL(3)$ Eisenstein series.

Let us introduce some notations first.

Notations. Let

$$w_1 = \begin{pmatrix} & & -1 \\ & 1 & \\ 1 & & \end{pmatrix}, \qquad w_2 = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}, \tag{3.5.1}$$

and for $\nu = (\nu_1, \nu_2) \in C^2$, set

$$\tilde{\nu} := (\nu_1, \nu_1), \quad \alpha := -\nu_1 - 2\nu_2 + 1, \quad \beta := -\nu_1 + \nu_2, \quad \gamma := 2\nu_1 + \nu_2 - 1. \tag{3.5.2}$$

For $s \in \mathbb{C}$, $k \in \mathbb{Z}$, Define

$$G(s,k,\nu) := \frac{\Gamma\left(\frac{1-s+2k+\alpha}{2}\right)\Gamma\left(\frac{1-s+2k+\beta}{2}\right)\Gamma\left(\frac{1-s+2k+\gamma}{2}\right)}{\Gamma\left(\frac{s-\alpha}{2}\right)\Gamma\left(\frac{s-\beta}{2}\right)\Gamma\left(\frac{s-\gamma}{2}\right)}. \tag{3.5.3}$$

Let f be a $GL(3)$ Maass form or an Eisenstein series of type $\nu = (\nu_1, \nu_2)$ for $SL(3,\mathbb{Z})$. We define the dual form $\tilde{f}$ by

$$\tilde{f}(z) := f(w_1 {}^t z^{-1} w_1). \tag{3.5.4}$$

Now we restrict to the case where f is a $GL(3)$ Maass form. Then f has Fourier expansion

$$f(z) \quad = \quad \sum_{\gamma \in U_2(\mathbb{Z})} \sum_{m_1=1}^{\infty} \frac{A(m_1,m_2)}{|m_1 m_2|} W_{\text{Jacquet}}\left(M\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, \nu, \Psi_{1,1}\right), \tag{3.5.5}$$

where $U_2(\mathbb{Z})$ is the group of unipotent 2×2 upper triangular matrices with coefficients in $\mathbb{Z}$, and $M = \mathrm{diag}(m_1|m_2|, m_1, 1)$.

Proposition 3.5.1. (The Voronoi formula for $GL(3)$ Maass forms)

Let f be a $GL(3)$ Hecke–Maass form of type $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$. Suppose $A(n,m)$ is the (n,m)-th Fourier coefficient of f. Let a, $\bar{a}$, c, $\delta \in \mathbb{Z}$ with $\delta > 0$, $c \neq 0$, $(a,c) = 1$, and $a\bar{a} \equiv 1 \pmod{c}$.

Let $\phi(x) \in C_c^\infty(0,\infty)$ be a test function, $\tilde{\phi}$ its Mellin transform, and

$$\begin{aligned}
\Phi_k(x) &:= \int_{\mathrm{Re}\, s_2=\sigma} (\pi^3 x)^{-s_2} G(-s_2, k, \nu)\tilde{\phi}(-s_2-k)\, ds_2, \\
\Phi_{0,1}^0(x) &:= \Phi_0(x) + \frac{\pi^{-3}c^3\delta^3}{m_1^2 m_2 i}\Phi_1(x), \\
\Phi_{0,1}^1(x) &:= \Phi_0(x) - \frac{\pi^{-3}c^3\delta^3}{m_1^2 m_2 i}\Phi_1(x).
\end{aligned} \tag{3.5.6}$$

Then

$$\begin{aligned}
&\sum_{m>0} A(\delta, m) e\left(\frac{m\bar{a}}{c}\right)\phi(m) \\
&= \frac{c\pi^{-\frac{5}{2}}}{4i} \sum_{m_1|c\delta} \sum_{m_2>0} \frac{A(m_2,m_1)}{m_1 m_2} S(\delta a, m_2; \delta c m_1^{-1}) \Phi_{0,1}^1\left(\frac{m_2 m_1^2}{c^3\delta}\right) \\
&\quad + \frac{c\pi^{-\frac{5}{2}}}{4i} \sum_{m_1|c\delta} \sum_{m_2>0} \frac{A(m_2,m_1)}{m_1 m_2} S(\delta a, -m_2; \delta c m_1^{-1}) \Phi_{0,1}^1\left(\frac{m_2 m_1^2}{c^3\delta}\right).
\end{aligned} \tag{3.5.7}$$

Proof. For details, see [14]. A sketch of the proof by Goldfeld and Li will be given below, to inspire the proof for the case of $GL(3)$ Eisenstein series. □

Proposition 3.5.2. (The Voronoi formula for $GL(3)$ Eisenstein series)

Let $\mathcal{E}_{min,\nu}(z)$ be a $GL(3)$ Hecke–Maass form of type $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$. Suppose $A_\nu(n,m)$ is the (n,m)-th Fourier coefficient of $\mathcal{E}_{min,\nu}(z)$. Let a, $\bar{a}$, c, $\delta \in \mathbb{Z}$ with $\delta > 0$, $c \neq 0$, $(a,c) = 1$, and $a\bar{a} \equiv 1 \pmod{c}$. Let $\nu_0 = \left(\frac{1}{3}, \frac{1}{3}\right)$.

Let $\phi(x) \in C_c^\infty(0,\infty)$ be a test function, $\tilde{\phi}$ its Mellin transform, and $\Phi_k(x)$, $\Phi_{0,1}^0(x)$, $\Phi_{0,1}^1(x)$ the *same as in (3.5.6). Then*

$$\begin{aligned}
&\sum_{m>0} A_{\nu_0}(\delta, m)\, e\left(\frac{m\bar{a}}{c}\right)\phi(m) \\
= \quad & \delta c \pi^{-\frac{3}{2}} \sum_{n|\delta c}\sum_{m>0} \frac{n^{-1}m^{-\frac{2}{3}}}{\xi(1)^3} \sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}} \sigma_{0,0}\left(\frac{n}{n_1 n_2}, m\right) S(m, \delta a; \delta c n^{-1}) \Phi_{0,1}^{0}\left(\frac{mn^2}{(\delta c)^3}\right) \\
& + \delta c \pi^{-\frac{3}{2}} \sum_{n|\delta c}\sum_{m>0} \frac{n^{-1}m^{-\frac{2}{3}}}{\xi(1)^3} \sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}} \sigma_{0,0}\left(\frac{n}{n_1 n_2}, m\right) S(-m, \delta a; \delta c n^{-1}) \Phi_{0,1}^{1}\left(\frac{mn^2}{(\delta c)^3}\right) \\
& + \frac{3\tilde{\phi}(1)}{\xi(1)\delta^2 c^2} \pi^{\frac{3}{2}} \Gamma^{-3}\left(\frac{1}{2}\right) \sum_{n|\delta c} n S(0, \delta a; \delta c n^{-1}) \sigma_0(n). \qquad (3.5.8)
\end{aligned}$$

Here $\sigma_{0,0}(n,m) := \sum_{\substack{d_1|m \\ d_1>0}} \sum_{\substack{d_2|\frac{m}{d_1} \\ d_2>0 \\ (d_2,n)=1}} 1$ *for* $n, m \in \mathbb{Z}$.

Remark. Compared to the Voronoi formula for Maass forms, the Voronoi formula for Eisenstein series contains an extra term. This comes from the residue of the Eisenstein series.

Before we start proving the Voronoi formula for Eisenstein series, we first sketch the proof for Maass forms by Goldfeld and Li [14].

Since f is automorphic, for any $z \in \mathfrak{h}^3$, we have

$$f(Auz) = \tilde{f}(w_2{}^t(Auz)^{-1}), \qquad (3.5.9)$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ \frac{h}{q} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad u = \begin{pmatrix} 1 & 0 & u_3 \\ & 1 & u_1 \\ & & 1 \end{pmatrix}. \qquad (3.5.10)$$

Denote

$$\delta := (h, q), \qquad h_\delta := \frac{h}{\delta}, \qquad q_\delta := \frac{q}{\delta}.$$

For $k = 0, 1$, let

$$F_k(y, h, q) := \left(\frac{\partial}{\partial x_2}\right)^k \int_0^1 \int_0^1 f(Auz)e(-qu_1)\, du_1\, du_3 \Big|_{x_1=x_2=0}, \tag{3.5.11}$$

with $y = \mathrm{diag}(y_1y_2, y_1, 1)$. Sometimes we also write it as $F_k(y_1, y_2, h, q)$.

For $\mathrm{Re}\, s_1$ large, define the "double Mellin transform" of $F_k(y, h, q)$ by

$$\tilde{F}_k(h, q, s) := \int_0^\infty \int_0^\infty F_k(y_1, y_2, h, q) y_1^{s_1-1} y_2^{s_2-2} \frac{dy_1}{y_1} \frac{dy_2}{y_2} \tag{3.5.12}$$

which can be proved to be absolutely convergent for all $s_2 \in \mathbb{C}$ and entire in s_2.

Furthermore, it has the following series expansion for $\mathrm{Re}\, s_2$ large and $-\mathrm{Re}\, s_2$ large, respectively:

Lemma 3.5.3. *For* $\mathrm{Re}\, s_2$ *large,*

$$\tilde{F}_k(h, q, s) = \frac{G_1(s_1, s_2, \nu)}{q_\delta^{s_1-2s_2+1}\delta^{s_1}} \sum_{m_2 \neq 0} \left(\frac{2\pi i m_2}{q_\delta^2}\right)^k \frac{A(\delta, m_2)}{|m_2|^{s_2}} e\left(\frac{m_2 \bar{h}_\delta}{q_\delta}\right) \tag{3.5.13}$$

with

$$G_1(s_1, s_2, \nu) = \int_0^\infty \int_0^\infty W_{Jacquet}(y, \nu, \psi_{1,1}) y_1^{s_1-1} y_2^{s_2-1} \frac{dy_1}{y_1} \frac{dy_2}{y_2}. \tag{3.5.14}$$

Lemma 3.5.4. *For* $-\mathrm{Re}\, s_2$ *large,*

$$\begin{aligned} \tilde{F}_k(h, q, s) \quad &= \quad G_2(s, \nu, k) \frac{(-2\pi i q)^k \pi^{\frac{s_1+s_2}{2}}}{q^{s_1+s_2}\Gamma\left(\frac{s_1+s_2}{2}\right)} \\ &\qquad \cdot \sum_{m_1 | q} \sum_{m_2 \neq 0} \left(\frac{i m_2}{|m_2|}\right)^k \frac{A(m_2, m_1)}{|m_1|^{2k+1-2s_2}|m_2|^{k+1-s_2}} S(h, m_2; q m_1^{-1}) \end{aligned} \tag{3.5.15}$$

with

$$\begin{aligned} G_2(s, \nu, k) \quad &= \quad \int_0^\infty \int_0^\infty W_{Jacquet}(y, \tilde{\nu}, \psi_{1,1}) K_{\frac{s_1+s_2-1-2k}{2}}(2\pi y_2) \\ &\qquad \cdot y_1^{2k+s_1-s_2} y_2^{\frac{2k+s_1-s_2-1}{2}} \frac{dy_1}{y_1} \frac{dy_2}{y_2}. \end{aligned} \tag{3.5.16}$$

Proof. These two lemmas follow from the modularity equation (3.5.9) and the Fourier expansions of $f(Auz)$ and $\tilde{f}(Auz)$, where $\tilde{f}(z)$ is the dual Maass form of f. For details, see [13]. $\square$

For $\mathrm{Re}\, s_2 > 3$, we take the "series parts" of (3.5.13) and (3.5.15), and define two series: (here $\bar{h}_\delta h_\delta \equiv 1 \pmod{q_\delta}$)

$$L_k(\bar{h}, q, s_2) \;\; \coloneqq \;\; \sum_{m_2>0} \frac{A(\delta, m_2)}{m_2^{s_2-k}} \left[e\left(\frac{m_2\bar{h}_\delta}{q_\delta}\right) + (-1)^k e\left(-\frac{m_2\bar{h}_\delta}{q_\delta}\right)\right], \tag{3.5.17}$$

$$\hat{L}_k(h, q, s_2) \;\; \coloneqq \;\; \sum_{m_1|q} \sum_{m_2>0} \frac{A(m_2, m_1)}{|m_1|^{2k+2s_2-1}|m_2|^{k+s_2}} \cdot [S(h, m_2; qm_1^{-1}) + (-1)^k S(h, -m_2; qm_1^{-1})]. \tag{3.5.18}$$

It follows from (3.5.13) and (3.5.15) that these series have analytic continuation to the whole complex plane and satisfy the functional equation:

$$L_k(\bar{h}, q, s_2) = \hat{L}_k(h, q, 1-s_2) i^{-k} q^{-3s_2+1+3k} \pi^{3s_2-3k-\frac{3}{2}} \delta^{2s_2-1-2k} G(s_2, k, \nu), \tag{3.5.19}$$

where

$$G(s_2, k, \nu) = \frac{\Gamma\left(\frac{1-s_2+2k+\alpha}{2}\right)\Gamma\left(\frac{1-s_2+2k+\beta}{2}\right)\Gamma\left(\frac{1-s_2+2k+\gamma}{2}\right)}{\Gamma\left(\frac{s_2-\alpha}{2}\right)\Gamma\left(\frac{s_2-\beta}{2}\right)\Gamma\left(\frac{s_2-\gamma}{2}\right)}. \tag{3.5.20}$$

Take a test function $\phi(x) \in C_c^\infty(0, \infty)$ and let $\tilde{\phi}(s) = \int_0^\infty \phi(x) x^s \, \frac{dx}{x}$ be its Mellin transform. For $\sigma > 3$, by Mellin inversion, we have

$$\begin{aligned} &\sum_{m_2\in\mathbb{Z}} A(\delta, m_2) \left[e\left(\frac{m_2\bar{h}_\delta}{q_\delta}\right) + (-1)^k e\left(-\frac{m_2\bar{h}_\delta}{q_\delta}\right)\right] \phi(m_2) \\ &\qquad = \frac{1}{2\pi i} \int_{\mathrm{Re}\, s_2=\sigma} \tilde{\phi}(s_2-k) L_k(\bar{h}, q, s_2)\, ds_2. \end{aligned} \tag{3.5.21}$$

Now moving the line of integration of the right side of (3.5.21) to $\mathrm{Re}\, s_2 = -\sigma$ and applying the functional equation (3.5.19), we obtain

$$\begin{aligned}
&\sum_{m_2\in\mathbb{Z}} A(\delta,m_2)\left[e\left(\frac{m_2\bar{h}_\delta}{q_\delta}\right)+(-1)^k e\left(-\frac{m_2\bar{h}_\delta}{q_\delta}\right)\right]\phi(m_2)\\
&\quad= \frac{1}{2\pi i}\int_{\mathrm{Re}\, s_2=-\sigma}\tilde{\phi}(s_2-k)L_k(\bar{h},q,s_2)\,ds_2\\
&\quad= \frac{1}{2\pi i}\int_{\mathrm{Re}\, s_2=-\sigma}\tilde{\phi}(s_2-k)\hat{L}_k(\bar{h},q,1-s_2)i^{-k}q^{-3s_2+1+3k}\pi^{3s_2-3k-\frac{3}{2}}\delta^{2s_2-1-2k}G(s_2,k,\nu)\,ds_2\\
&\quad\overset{(s_2\mapsto -s_2)}{=} -\frac{q^{1+3k}}{2\pi^{3k+\frac{5}{2}}i^{1+k}\delta^{1+2k}}\int_{\mathrm{Re}\, s_2=\sigma}\tilde{\phi}(-s_2-k)\hat{L}_k(\bar{h},q,1+s_2)q^{3s_2}\pi^{-3s_2}\delta^{-2s_2}G(-s_2,k,\nu)\,ds_2\\
&\quad\overset{(3.5.18)}{=} -\frac{q^{1+3k}}{2\pi^{3k+\frac{5}{2}}i^{1+k}\delta^{1+2k}}\int_{\mathrm{Re}\, s_2=\sigma}\tilde{\phi}(-s_2-k)q^{3s_2}\pi^{-3s_2}\delta^{-2s_2}G(-s_2,k,\nu)\\
&\qquad\qquad\cdot\sum_{m_1|q}\sum_{m_2>0}\frac{A(m_2,m_1)}{|m_1|^{2k+2s_2+1}|m_2|^{k+s_2+1}}\cdot\left[S(h,m_2;qm_1^{-1})+(-1)^k S(h,-m_2;qm_1^{-1})\right]ds_2\\
&\quad= -\frac{q_\delta^{1+3k}\delta^k}{2\pi^{3k+\frac{5}{2}}i^{1+k}}\sum_{m_1|q_\delta\delta}\sum_{m_2>0}\frac{A(m_2,m_1)}{|m_1|^{2k+1}|m_2|^{k+1}}\\
&\qquad\qquad\cdot\left[S(\delta h_\delta,m_2;\delta q_\delta m_1^{-1})+(-1)^k S(\delta h_\delta,-m_2;\delta q_\delta m_1^{-1})\right]\Phi_k\left(\frac{m_2 m_1}{q_\delta^3\delta}\right)
\end{aligned}$$

where

$$\Phi_k(x)=\int_{\mathrm{Re}\, s_2=\sigma}(\pi^3 x)^{-s_2}G(-s_2,k,\nu)\tilde{\phi}(-s_2-k)\,ds_2.$$

Recall that $\delta=(h,q)$, $h=\delta h_\delta$ and $q=\delta q_\delta$. Now we set $a=h_\delta$ and $c=q_\delta$. Hence $(a,c)=1$. To write the formula in a neater way, let

$$\Phi_{0,1}^0(x)=\Phi_0(x)+\frac{\pi^{-3}c^3\delta}{m_1^2 m_2 i}\Phi_1(x),\qquad \Phi_{0,1}^1(x)=\Phi_0(x)-\frac{\pi^{-3}c^3\delta}{m_1^2 m_2 i}\Phi_1(x).$$

Then we obtain the Voronoi formula for $GL(3)$ Maass forms (Proposition 3.5.1) as desired.

The proof for the Voronoi formula for an Eisenstein series is similar to that for a Maass form. Instead of doing Fourier analysis on $\mathscr{E}_{\min,\nu}(z)$, one does that only for the non-degenerate terms of $\mathscr{E}_{\min,\nu}(z)$. The L-functions thus constructed will have poles and hence extra residue terms will appear in the Voronoi formula. Let us elaborate on this as follows.

For all ν_1 and ν_2, except those on the complex lines $\nu_1=\frac{2}{3},\frac{1}{3}\rho_1$ or $\nu_2=\frac{2}{3},\frac{1}{3}\rho_2$ where ρ_1,ρ_2 run through all nontrivial zeros of $\zeta(s)$, we have the Fourier expansion of $\mathscr{E}_{\min,\nu}(z)$ where each term can be written in the form of (c.f. [46])

$$\varphi(y) \qquad \text{or} \qquad \varphi(y)\cdot(\text{K-Bessel function})\cdot(\text{a character of } x).$$

When the character of x involves all x_1, x_2, x_3, we say that the term is non-degenerate. Denote the sum of all non-degenerate terms by $\mathscr{E}^{\mathrm{nondeg}}_{\min,\nu}(z)$. Then

$$\begin{aligned}
\mathscr{E}^{\mathrm{nondeg}}_{\min,\nu}(z) \quad = \quad & \frac{2}{\xi(3\nu_1)\xi(3\nu_2)\xi(3\nu_1+3\nu_2-1)} \sum_{(c,d)=1}\sum_{n\geq 1}\sum_{m\neq 0}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}} \\
& \cdot n^{-2+\nu_1+2\nu_2}|m|^{-2+2\nu_1+\nu_2} n_1^{-3\nu_1-3\nu_2+2} n_2^{1-3\nu_2} \sigma_{1-3\nu_1,2-3\nu_1-3\nu_2}\left(\frac{n}{n_1 n_2}, |m|\right) \\
& \cdot W_{\mathrm{Jacquet}}\left(\begin{pmatrix} \frac{n|m|y_1y_2}{|cz_2+d|} & & \\ & ny_1|cz_2+d| & \\ & & 1\end{pmatrix}, \nu\right) e\left(n(cx_3+dx_1) + m\,\mathrm{Re}\,\frac{az_2+b}{cz_2+d}\right).
\end{aligned} \tag{3.5.22}$$

Remark. To be precise, the Fourier expansion of $\mathscr{E}_{\min,\nu}(z)$ can be written explicitly as follows:

$$\mathscr{E}_{\min,\nu}(z) = (\text{degenerate terms}) + (\text{non-degenerate terms}),$$

where

$$(\text{degenerate terms}) = \mathscr{E}^{0}_{\min,\nu}(z) + \mathscr{E}^{11}_{\min,\nu}(z) + \mathscr{E}^{12}_{\min,\nu}(z),$$

and

$$\begin{aligned}
\mathscr{E}^{0}_{\min,\nu}(z) &= y_1^{2\nu_1+\nu_2} y_2^{\nu_1+2\nu_2} \\
&\quad + c\left(\frac{3\nu_1}{2}\right) y_1^{1-\nu_1+\nu_2} y_2^{\nu_1+2\nu_2} + c\left(\frac{3\nu_2}{2}\right) y_2^{1+\nu_1-\nu_2} y_1^{2\nu_1+\nu_2} \\
&\quad + c\left(\frac{3\nu_1}{2}\right) c\left(\frac{3\nu_1+3\nu_2-1}{2}\right) y_1^{1+\nu_2-\nu_1} y_2^{-\nu_2-2\nu_1+3} \\
&\quad + c\left(\frac{3\nu_2}{2}\right) c\left(\frac{3\nu_1+3\nu_2-1}{2}\right) y_1^{2-\nu_1-2\nu_2} y_2^{1+\nu_1-\nu_2} \\
&\quad + c\left(\frac{3\nu_1}{2}\right) c\left(\frac{3\nu_2}{2}\right) c\left(\frac{3\nu_1+3\nu_2-1}{2}\right) y_1^{2-\nu_1-2\nu_2} y_2^{2-2\nu_1-\nu_2},
\end{aligned}$$

$$\begin{aligned}
\mathscr{E}^{11}_{\min,\nu}(z) &= \frac{2(y_1^2y_2)^{1-\frac{\nu_1}{2}-\nu_2} y_2^{\frac{1}{2}}}{\xi(3\nu_1)} c\left(\frac{3\nu_2}{2}\right) c\left(\frac{3\nu_1+3\nu_2-1}{2}\right) \sum_{m\neq 0} |m|^{\frac{3\nu_1}{2}-\frac{1}{2}} \sigma_{1-3\nu_1}(|m|) K_{\frac{3\nu_1-1}{2}}(2\pi|m|y_2) e(mx_2) \\
&\quad + \frac{2(y_1^2y_2)^{\frac{1}{2}+\frac{3}{4}(\nu_2-\nu_1)} y_2^{\frac{1}{2}}}{\xi(3\nu_2)} c\left(\frac{3\nu_1}{2}\right) \sum_{m\neq 0} |m|^{\frac{3\nu_1+3\nu_2-2}{2}} \sigma_{4-3\nu_1-3\nu_2}(|m|) K_{\frac{3\nu_1+3\nu_2-2}{2}}(2\pi|m|y_2) e(mx_2), \\
&\quad + \frac{2(y_1^2y_2)^{1-\frac{\nu_1}{2}-\nu_2} y_2^{\frac{1}{2}}}{\xi(3\nu_1)} \sum_{m\neq 0} |m|^{\frac{3\nu_2}{2}-\frac{1}{2}} \sigma_{1-3\nu_2}(|m|) K_{\frac{3\nu_2-1}{2}}(2\pi|m|y_2) e(mx_2)
\end{aligned}$$

and

$$\begin{aligned}
\mathscr{E}^{12}_{\min,\nu}(z) &= \frac{2(y_1^2y_2)^{\frac{1}{4}+\frac{1}{4}\nu_1+\frac{1}{2}\nu_2}}{\xi(3\nu_1)} \sum_{(c,d)=1} \sum_{n\leq 1} n^{\frac{3}{2}\nu_1-1} \sigma_{1-3\nu_1} \\
&\qquad \cdot u_{c,d}^{\frac{3}{4}\nu_1+\frac{3}{2}\nu_2-\frac{1}{4}} K_{\frac{3\nu_1-1}{2}}\left(2\pi n\sqrt{\frac{y_1^2y_2}{u_{c,d}}}\right) e\left((cx_3+dx_1)n\right), \\
&\quad + \frac{2(y_1^2y_2)^{\frac{3}{4}-\frac{1}{2}\nu_1-\frac{1}{4}\nu_2}}{\xi(3\nu_2)} c\left(\frac{3\nu_1}{2}\right) c\left(\frac{3\nu_1+3\nu_2-1}{2}\right) \sum_{(c,d)=1} \sum_{n\geq 1} \\
&\qquad \cdot n^{\frac{3}{2}-\frac{1}{2}} \sigma_{1-3\nu_2}(n) u_{c,d}^{\frac{5}{4}-\frac{3}{2}\nu_1-\frac{3}{4}\nu_2} K_{\frac{3\nu_2-1}{2}}\left(2\pi n\sqrt{\frac{y_1^2y_2}{u_{c,d}}}\right) e\left((cx_3+dx_1)n\right), \\
&\quad + \frac{2(y_1^2y_2)^{\frac{1}{2}+\frac{1}{4}(\nu_1-\nu_2)}}{\xi(3\nu_1+3\nu_2-1)} c\left(\frac{3\nu_2}{2}\right) \sum_{(c,d)=1} \sum_{n\geq 1} n^{\frac{3\nu_1+3\nu_2-1}{2}} \\
&\qquad \cdot \sigma_{-3\nu_1-3\nu_2+2}(n) u_{c,d}^{\frac{1}{2}+\frac{3}{4}\nu_1-\frac{3}{4}\nu_2} K_{\frac{3\nu_1+3\nu_2-2}{2}}\left(2\pi n\sqrt{\frac{y_1^2y_2}{u_{c,d}}}\right) e\left((cx_3+dx_1)n\right).
\end{aligned}$$

The non-degenerate term $\mathscr{E}^{\mathrm{nondeg}}_{\min,\nu}(z)$ is given above explicitly.

For $k = 0, 1$, let

$$F_k(y,\nu) := \left(\frac{\partial}{\partial x_2}\right)^k \int_0^1 \int_0^1 \mathscr{E}^{\text{nondeg}}_{\min,\nu}(Auz)e(-qu_1)\,du_1\,du_3\bigg|_{x_1=x_2=0}, \tag{3.5.23}$$

with $y = \operatorname{diag}(y_1y_2, y_1, 1)$.

Notice that (3.5.23) is an analogue of (3.5.11) but here we use the non-degenerate part of the Eisenstein series instead.

From (3.5.22) and (2.1), one can see that $F_k(y,\nu)$ defined above has rapid decay as $y_2 \to \infty$, so the following integration can be defined for $\operatorname{Re}(w_2)$ large and for $k = 0, 1$:

$$L_k(y_1, w_2, \nu) := \int_0^\infty F_k(y,\nu)y_2^{w_2-1}\,\frac{dy_2}{y_2}. \tag{3.5.24}$$

This is the Mellin transform of $F_k(y,\nu)$ with respect to y_2.

Lemma 3.5.5. *Let $L_0(y_1, w_2, \nu)$ and $L_1(y_1, w_2, \nu)$ be defined as above. Fix $y_1 > 0$, $\nu \in \mathbb{C}^2$, then*

(i) $L_1(y_1, w_2, \nu)$ is entire as a complex function of w_2;

(ii) $L_0(y_1, w_2, \nu)$, as a complex function of w_2, has a meromorphic continuation to $\mathbb{C}$ with 6 simple poles $w_2 = 2 - \nu_1 - 2\nu_2$, $2\nu_1 + \nu_2$, $1 + \nu_2 - \nu_1$, $1 - \nu_1 - 2\nu_2$, $2\nu_1 + \nu_2 - 1$ and $\nu_2 - \nu_1$; unless we are in the cases that some of the poles coincide.

Proof. We break the integral into two parts:

$$L_k(y_1, w_2, \nu) = \int_0^1 F_k(y,\nu)y_2^{w_2-1}\,\frac{dy_2}{y_2} + \int_1^\infty F_k(y,\nu)y_2^{w_2-1}\,\frac{dy_2}{y_2}.$$

Since $\int_1^\infty F_k(y,\nu)y_2^{w_2-1}\,\frac{dy_2}{y_2}$ is entire in w_2, it suffices to consider the first term.

It is easy to check that the Eisenstein series $\mathscr{E}_{\min,\nu}(z)$ satisfies the following modularity relation:

$$\tilde{\mathscr{E}}_{\min,\tilde{\nu}}(w_2{}^t(Auz)^{-1}) = \mathscr{E}_{\min,\nu}(Auz) \tag{3.5.25}$$

where $\tilde{\mathscr{E}}_{\min,\nu}(z) := \mathscr{E}_{\min,\nu}(w_1{}^tz^{-1}w_1)$ is the dual Eisenstein series.

Recall that $\mathscr{E}^{\text{nondeg}}_{\min,\nu} = \mathscr{E}_{\min,\nu} - \mathscr{E}^{0}_{\min,\nu} - \mathscr{E}^{11}_{\min,\nu} - \mathscr{E}^{12}_{\min,\nu}$. Together with (3.5.25), we have

$$\int_0^1 F_k(y,\nu) y_2^{w_2-1}\,\frac{dy_2}{y_2} = \int_0^1 (F_{k,1}(y,\nu) + F_{k,2}(y,\nu)) y_2^{w_2-1}\,\frac{dy_2}{y_2}$$

where

$$\begin{aligned} F_{k,1}(y,\nu) &:= \left(\frac{\partial}{\partial x_2}\right)^k \int_0^1 \int_0^1 \mathscr{E}^{\text{nondeg}}_{\min,\tilde{\nu}}\left(w_2{}^t(Auz)^{-1}\right) e(-qu_1)\,du_1 du_3\Big|_{x_1=x_2=0} \\ &= \left(\frac{\partial}{\partial x_2}\right)^k \int_0^1 \int_0^1 \left(E_{\tilde{\nu}} - E^0_{\tilde{\nu}} - E^{11}_{\tilde{\nu}} - E^{12}_{\tilde{\nu}}\right)\left(w_2{}^t(Auz)^{-1}\right) e(-qu_1)\,du_1 du_3\Big|_{x_1=x_2=0} \end{aligned}$$

which has rapid decay as $y_2 \to 0$, and hence $\int_0^1 F_{k,1}(y,\nu) y_2^{w_2-1}\frac{dy_2}{y_2}$ is holomorphic for all $w_2 \in \mathbb{C}$; and

$$\begin{aligned} F_{k,2}(y,\nu) &:= \left(\frac{\partial}{\partial x_2}\right)^k \int_0^1 \int_0^1 \left[\left(\mathscr{E}^{0}_{\min,\nu} + \mathscr{E}^{11}_{\min,\nu} + \mathscr{E}^{12}_{\min,\nu}\right)\left(w_2{}^t(Auz)^{-1}, \tilde{\nu}\right) - \left(\mathscr{E}^{0}_{\min,\nu} + \mathscr{E}^{11}_{\min,\nu} + \mathscr{E}^{12}_{\min,\nu}\right)(Auz,\nu)\right] \\ &\qquad \cdot e(-qu_1)\,du_1 du_3\Big|_{x_1=x_2=0}. \end{aligned}$$

Direct computation shows that the contribution from $\mathscr{E}^{0}_{\min,\nu}$ and $\mathscr{E}^{11}_{\min,\nu}$ to $F_{k,1}(y,\nu)$ is 0. Therefore,

$$F_{k,2}(y,\nu) = \left(\frac{\partial}{\partial x_2}\right)^k \int_0^1 \int_0^1 \left[E^{12}_{\tilde{\nu}}\left(w_2{}^t(Auz)^{-1}\right) - \mathscr{E}^{12}_{\min,\nu}(Auz)\right] \cdot e(-qu_1)\,du_1 du_3\Big|_{x_1=x_2=0}.$$

Now we discuss $F_{k_2}(y,\nu)$ for the cases $k=0$ and $k=1$ separately:

- When $k = 0$:

  For $\operatorname{Re} w_2$ large, by direct computation one gets

$$\begin{aligned} \int_0^1 F_{0,2}(y_1,w_2,\nu) y_2^{w_2-1}\frac{dy_2}{y_2} &= G_1(y_1,\nu)/(w_2+2\nu_2+\nu_1-2) + G_2(y_1,\nu)/(w_2-\nu_2-2\nu_1) \\ &\quad + G_3(y_1,\nu)/(w_2-\nu_2+\nu_1-1) + G_4(y_1,\nu)/(w_2+2\nu_2+\nu_1-1) \\ &\quad + G_5(y_1,\nu)/(w_2-2\nu_2-\nu_1+1) + G_6(y_1,\nu)/(w_2-\nu_2+\nu_1), \end{aligned}$$

  where $G_i(y_1,\nu)$ are functions which do not involve w_2. The expressions of $G_i(y_1,\nu)$ can be explicitly written down, but since they are complicated and irrelevant, let us omit them here.

Therefore, the integration above, viewed as a complex function in the variable w_2, has meromorphic continuation with poles at $w_2 = 2\nu_1 + \nu_2$, $2 - \nu_1 - 2\nu_2$, $1 + \nu_2 - \nu_1$, $1 - \nu_1 - 2\nu_2$, $2\nu_1 + \nu_2 - 1$ and $\nu_2 - \nu_1$. These are exactly the poles of $L_0(y, w_2, \nu)$.

- When $k = 1$: by direct computation one can prove that $\int_0^1 F_{1,2}(y_1, w_2, \nu) y_2^{w_2-1} \frac{dy_2}{y_2} = 0$.

Combining the results, we see that $L_1(y, w_2, \nu)$ is an entire function of w_2. □

Now we further take the Mellin transform with respect to y_1. To simplify notation, we will write $w = (w_1, w_2)$. For $\mathrm{Re}(w_1)$ large, one defines

$$\Lambda_k(w, \nu) := \int_0^\infty L_k(y_1, w_2, \nu) y_1^{w_1-1} \frac{dy_1}{y_1}. \tag{3.5.26}$$

Similar to the case of Maass forms, $\Lambda_k(w, \nu)$ has series expansion as follows:

Lemma 3.5.6. *For $k = 0, 1$, $\mathrm{Re}(w_2)$ large, we have*

$$\Lambda_k(w, \nu) = \frac{1}{q_\delta^{w_1-2w_2+1}\delta^{w_1}} \sum_{m_2 \neq 0} \frac{A_\nu(\delta, m_2)}{|m_2|^{w_2}} \left(\frac{2\pi i m_2}{q_\delta^2}\right)^k e\left(\frac{m_2 \bar{h}_\delta}{q_\delta}\right) G_1(w, \nu) \tag{3.5.27}$$

with

$$G_1(w, \nu) = \int_0^\infty \int_0^\infty W_{Jacquet}(y, \nu, \psi_{1,1}) y_1^{w_1-1} y_2^{w_2-1} \frac{dy_1}{y_1} \frac{dy_2}{y_2}. \tag{3.5.28}$$

Lemma 3.5.7. *For $-\mathrm{Re}(w_2)$ large,*

$$\begin{aligned} \Lambda_k(w, \nu) \quad = \quad & \frac{(-2\pi i q)^k \pi^{\frac{w_1+w_2}{2}}}{q^{w_1+w_2}\Gamma\left(\frac{w_1+w_2}{2}\right)} \sum_{m_1 | q} \sum_{m_2 \neq 0} \frac{A_\nu(m_2, m_1)}{|m_1|^{2k+1-2w_2}|m_2|^{k+1-w_2}} \left(\frac{i m_2}{|m_2|}\right)^k \\ & \cdot S(h, m_2; q m_1^{-1}) G_2(w, \nu, k) \end{aligned} \tag{3.5.29}$$

with

$$\begin{aligned} G_2(w, \tilde{\nu}, k) \quad = \quad & \int_0^\infty \int_0^\infty W_{Jacquet}(y, \tilde{\nu}, \psi_{1,1}) K_{\frac{w_1+w_2-1-2k}{2}}(2\pi y_2) \\ & \cdot y_1^{2k+s_1-w_2} y_2^{\frac{2k+w_1-w_2-1}{2}} \frac{dy_1}{y_1} \frac{dy_2}{y_2}. \end{aligned} \tag{3.5.30}$$

Remark. By [3],

$$G_1(w,\nu) = \pi^{\frac{-w_1-w_2}{4}} \frac{\Gamma\left(\frac{w_1+\alpha}{2}\right)\Gamma\left(\frac{w_1+\beta}{2}\right)\Gamma\left(\frac{w_1+\gamma}{2}\right)\Gamma\left(\frac{w_2-\alpha}{2}\right)\Gamma\left(\frac{w_2-\beta}{2}\right)\Gamma\left(\frac{w_2-\gamma}{2}\right)}{\Gamma\left(\frac{w_1+w_2}{2}\right)},$$

for $\mathrm{Re}(w_1)$ and $\mathrm{Re}(w_2)$ large. Recall that α, β, γ are defined in (3.5.2).

By [16],

$$\begin{aligned} G_2(w,\tilde{\nu},k) &= \frac{1}{4}\pi^{\frac{-3w_1+3w_2-6k-3}{2}}\Gamma\left(\frac{w_1+\alpha}{2}\right)\Gamma\left(\frac{w_1+\beta}{2}\right)\Gamma\left(\frac{w_1+\gamma}{2}\right) \\ &\quad \cdot\Gamma\left(\frac{1-w_2+2k+\alpha}{2}\right)\Gamma\left(\frac{1-w_2+2k+\beta}{2}\right)\Gamma\left(\frac{1-w_2+2k+\gamma}{2}\right). \end{aligned}$$

Proof. (i) For Lemma 3.5.7: Recall the notations in (3.5.10).

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ h/q & 1 \end{pmatrix} = \begin{pmatrix} a+bh/q & b \\ c+dh/q & d \end{pmatrix} = \begin{pmatrix} a' & b \\ c' & d \end{pmatrix}$$

By (3.5.22), we have

$$\begin{aligned} &\int_0^1\int_0^1 \mathscr{E}^{\mathrm{nondeg}}_{\min,\nu}(Auz)e(-qu_1)\,du_1du_3 \\ &= \sum_{(c,d)=1}\sum_{n=1}^{\infty}\sum_{m\neq 0}\frac{A_\nu(n,m)}{n|m|}e\left(m\,\mathrm{Re}\,\frac{a'z_2+b}{c'z_2+b}\right) \\ &\quad \cdot W_{\mathrm{Jacquet}}\left(\begin{pmatrix} \frac{n|m|y_1y_2}{|c'z_2+d|} & & \\ & ny_1\,|c'z_2+d| & \\ & & 1 \end{pmatrix},\nu\right) \\ &\quad \cdot\int_0^1\int_0^1 e(nc'(x_3+u_3)+nd(x_1+u_1))e(-qu_1)\,du_1du_3. \end{aligned} \tag{3.5.31}$$

Here $z_2 = x_2 + iy_2$.

The integral over u_1 and u_3 vanishes only when $nd = q$ and $nc' = n\left(c+\frac{dh}{q}\right) = 0$. Together with the condition $(c,d)=1$, these imply that $n=\delta=(h,q)$. Let $h_\delta = h/\delta$, $q_\delta = q/\delta$. Then

$$\int_0^1\int_0^1 \mathscr{E}^{\mathrm{nondeg}}_{\min,\nu}(Auz)e(-qu_1)\,du_1du_3$$
$$= \sum_{(c,d)=1}\sum_{m\neq 0}\frac{A_\nu(\delta,m)}{\delta|m|}e\left(m\,\mathrm{Re}\,\frac{a'z_2+b}{c'z_2+b}\right)W_{\mathrm{Jacquet}}\left(\begin{pmatrix}\frac{\delta|m|y_1y_2}{|c'z_2+d|} & & \\ & \delta y_1\,|c'z_2+d| & \\ & & 1\end{pmatrix},\nu\right)$$

Now Lemma 3.5.6 easily follows.

(ii) For Lemma 3.5.7: Recall that in the proof of Lemma 3.5.5, we have defined $F_{k,1}$ and $F_{k,2}$. Further

$$\begin{aligned}\Lambda_k(w,\nu) &= \int_0^\infty L_k(y_1,w_2,\nu)y_1^{w_1-1}\frac{dy_1}{y_1}\\ &= \int_0^\infty\int_0^\infty \Big[F_{k,1}(y,\nu)+F_{k,2}(y,\nu)\Big]y_1^{w_1-1}y_2^{w_2-1}\frac{dy_1}{y_1}\,\frac{dy_2}{y_2}.\end{aligned}$$

Direct computation shows that when $-\mathrm{Re}(w_2)$ is large,

$$\int_1^\infty F_{k,2}(y,\nu)y_2^{w_2-1}\frac{dy_2}{y_2} = -\int_0^1 F_{k,2}(y,\nu)y_2^{w_2-1}\frac{dy_2}{y_2}.$$

Therefore,

$$\Lambda_k(w,\nu) \quad = \int_0^\infty\int_0^\infty F_{k,1}(y,\nu)y_1^{w_1-1}y_2^{w_2-1}\frac{dy_1}{y_1}\,\frac{dy_2}{y_2}.$$

Recall that

$$F_{k,1}(y,\nu) = \left(\frac{\partial}{\partial x_2}\right)^k\int_0^1\int_0^1 \mathscr{E}^{\mathrm{nondeg}}_{\min,\tilde{\nu}}\left(w_2{}^t(Auz)^{-1}\right)e(-qu_1)\,du_1du_3\Big|_{x_1=x_2=0}.$$

By direct computation, we have

$$\int_0^1 \int_0^1 \mathscr{E}^{\text{nondeg}}_{\min,\tilde{\nu}}(w_2{}^t(Auz)^{-1})e(-qu_2)\,du_1du_3$$
$$= \sum_{(c,d)=1} \sum_{n=1}^{\infty} \sum_{m\neq 0} \frac{A_{\tilde{\nu}}(n,m)}{n|m|} W_{\text{Jacquet}}\left(\begin{pmatrix} \frac{n|m|y_1y_2}{|cz_2'+d|} & & \\ & \frac{ny_2|cz_2'+d|}{|z_2|^2} & \\ & & 1 \end{pmatrix}, \tilde{\nu}\right)$$
$$\cdot \int_0^1 \int_0^1 e\left(-ncu_1 - ncx_1 + \frac{ncx_2(u_3+x_3)}{|z_2|^2} - \frac{ndh}{q} - \frac{ndx_2}{|z_2|^2} - qu_1 + m\,\mathrm{Re}\,\frac{az_2'+b}{cz_2'+d}\right)du_1du_3.$$

Here $z_2 := x_2 + iy_2$ and $z_2' := -u_3 - x_3 + iy_1\,|z_2|$.

The u_1 integral vanishes unless $nc + q = 0$.

Note that $\frac{az+d}{cz+d} = \frac{a}{c} - \frac{1}{c(cz+d)}$. Since $(c,d) = 1$, we may write $d = lc + r$ for some $l \in \mathbb{Z}$ and $1 \leq r < |c|$ with $(r,c) = 1$. After changing variables $u_3 \mapsto u_3 + l - \frac{rn}{q}$, we have

$$\int_0^1 \mathscr{E}^{\text{nondeg}}_{\min,\tilde{\nu}}(w_2{}^t(Auz)^{-1})e(-qu_2)\,du_1du_3$$
$$= e(qx_1) \sum_{n|q} \sum_{m\neq 0} \frac{A_{\tilde{\nu}}(n,m)}{n|m|} S(h,m;qn^{-1}) W_{\text{Jacquet}}\left(\begin{pmatrix} \frac{n|m|y_1y_2}{|z_2|\cdot|cz_2'+d|} & & \\ & \frac{ny_2|cz_2'|}{|z_2|^2} & \\ & & 1 \end{pmatrix}, \tilde{\nu}\right)$$
$$\cdot \sum_{l\in\mathbb{Z}} \int_{l-\frac{rn}{q}}^{1+l-\frac{rn}{q}} e\left(\frac{-qx_2(u_3+x_3)}{|z_2|^2} - m\,\mathrm{Re}\,\frac{1}{c^2z_2'}\right)du_3.$$

Now we change variables successively:

$$u_3 \mapsto u_3 - x_3, \qquad u_3 \mapsto u_3y_1|z_2|,$$

then the above becomes

$$
\begin{aligned}
&\int_0^1 \mathscr{E}^{\text{nondeg}}_{\min,\tilde{\nu}}(w_2{}^t(Auz)^{-1})e(-qu_2)\,du_1du_3 \\
&= \quad e(qx_1)y_1|z_2|\sum_{n|q}\sum_{m\neq 0}\frac{A_{\tilde{\nu}}(n,m)}{n|m|}S(h,m;qn^{-1})W_{\text{Jacquet}}\left(\begin{pmatrix} \frac{n^2|m|y_2}{q|z_2|\sqrt{u_3^2+1}} & & \\ & \frac{qy_1y_2\sqrt{u_3^2+1}}{|z_2|} & \\ & & 1\end{pmatrix},\tilde{\nu}\right) \\
&\qquad\qquad \cdot\int_{-\infty}^{\infty} e\left(\frac{-qx_2y_1u_3)}{|z_2|}+\frac{n^2mu_3}{q^2y_1|z_2|(u_3^2+1)}\right)du_3.
\end{aligned}
$$

Taking partial derivatives with respect to x_2 and setting $x_1 = x_2 = 0$, we get

$$
\begin{aligned}
F_{k_1}(y,\nu) \quad = \quad & y_1y_2\sum_{n|q}\sum_{m\neq 0}\frac{A_{\tilde{\nu}}(n,m)}{n|m|}S(h,m;qn^{-1}) \\
& \cdot\int_{-\infty}^{\infty}\left(\frac{-2\pi iqy_1u_3}{y_2}\right)^k e\left(\frac{n^2mu_3}{q^2y_1y_2(u_3^2+1)}\right) \\
& \cdot W_{\text{Jacquet}}\left(\begin{pmatrix} \frac{n^2|m|}{qy_2\sqrt{u_3^2+1}} & & \\ & qy_2\sqrt{u_3^2+1} & \\ & & 1\end{pmatrix},\nu\right)du_3.
\end{aligned}
$$

Finally, make the transformations

$$
y_1 \mapsto \frac{y_1}{q\sqrt{u_3^2+1}}, \qquad y_2 \mapsto \frac{n^2|m|}{q\sqrt{u_3^2+1}y_1y_2},
$$

and apply the formulas [47] below for $\operatorname{Re} s > 0$:

$$
\begin{aligned}
\int_{-\infty}^{\infty} e(uy_2)(u^2+1)^{-s}\,du \quad &= \quad \frac{2\pi^s}{\Gamma(s)}|y_2|^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|y_2|), \\
\int_{-\infty}^{\infty} e(uy_2)(u^2+1)^{-s}u\,du \quad &= \quad \frac{2\pi^s}{\Gamma(s)}iy_2|y_2|^{s-\frac{3}{2}}K_{s-\frac{3}{2}}(2\pi|y_2|).
\end{aligned}
$$

We obtain Lemma 3.5.7 as desired.

□

For $k = 0, 1$, $\mathrm{Re}(w_2)$ being large, we define

$$L_k(\bar{h}, q, w_2) := q^{w_1-2w_2+1+2k}(2\pi i)^{-k} G_1^{-1}(w, \nu)\Lambda_k(w, \nu). \tag{3.5.32}$$

Though, a priori, the expression on the right side involves w_1, it does not actually depend on w_1. In fact, by Lemma 3.5.6 and Lemma 3.5.7, we have

$$L_k(\bar{h}, q, w_2) = \sum_{m>0} \frac{A_\nu(\delta, m_2)}{|m_2|^{w_2-k}} \left[e\left(\frac{m\bar{h}_\delta}{q_\delta}\right) + (-1)^k e\left(-\frac{m\bar{h}_\delta}{q_\delta}\right)\right] \tag{3.5.33}$$

for $\mathrm{Re}(w_2)$ large.

Then $L_k(\bar{h}, q, w_2)$ inherits analytic properties from $\Lambda_k(w, \nu)$. It is easy to check that:

- $L_1(\bar{h}, q, w_2)$ has analytic continuation to $\mathbb{C}$;
- $L_0(\bar{h}, q, w_2)$ has meromorphic continuation to $\mathbb{C}$ with poles at $w_2 = 2 - \nu_1 - 2\nu_2$, $2\nu_1 + \nu_2$, $1 + \nu_2 - \nu_1$, $1 - \nu_1 - 2\nu_2$, $2\nu_1 + \nu_2 - 1$ and $\nu_2 - \nu_1$; unless in the cases that some of the poles coincide.

  With the following two formulas [16] one can easily compute the residues explicitly:

$$\begin{aligned} \int_0^\infty y^s K_\nu(y) \frac{dy}{y} &= 2^{s-2}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right), && \text{if } \mathrm{Re}(s) > |\mathrm{Re}\,\nu|, \\ \int_{-\infty}^\infty (u_3^s + 1)^{-w}\, dw &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(w - \frac{1}{2}\right)}{\Gamma(w)}, && \text{if } \mathrm{Re}(w) > \frac{1}{2}. \end{aligned}$$

  The residues are listed as follows:

  I) The residue of $L_0(\bar{h}, q, w_2)$ at $w_2 = -\nu_1 - 2\nu_2 + 2$ is:

$$\begin{aligned} &2\xi(3\nu_1) q^{6\nu_2+3\nu_1-5} \pi^{-3\nu_2 - \frac{3\nu_1}{2} + \frac{5}{2}} \sum_{n|q} n^{-3\nu_2+2} S(0, h; qn^{-1}) \sigma_{1-3\nu_1}(n) \\ &\quad \cdot c\left(\frac{3\nu_2}{2}\right) c\left(\frac{3\nu_1 + 3\nu_2 - 1}{2}\right) \Gamma^{-1}\left(\frac{-3\nu_2 + 2}{2}\right) \Gamma^{-1}\left(\frac{-3\nu_1 - 3\nu_2 + 3}{2}\right); \end{aligned}$$

II) The residue of $L_0(\bar{h}, q, w_2)$ at $w_2 = 2\nu_1 + \nu_2$ is:

$$2\xi(3\nu_2)q^{-6\nu_2-3\nu_1+1}\pi^{3\nu_1+\frac{3\nu_2}{2}-\frac{1}{2}} \cdot \sum_{n|q} n^{3\nu_1+3\nu_2-1} S(0, h; qn^{-1})\sigma_{1-3\nu_2}(n) \cdot \Gamma^{-1}\left(\frac{3\nu_1}{2}\right)\Gamma^{-1}\left(\frac{3\nu_1+3\nu_2-1}{2}\right);$$

III) The residue of $L_0(\bar{h}, q, w_2)$ at $w_2 = \nu_2 - \nu_1 + 1$ is:

$$2\xi(3\nu_1+3\nu_2-1)q^{3\nu_1-3\nu_2-2}\pi^{\frac{3\nu_2}{2}-\frac{3\nu_1}{2}+1} \cdot \sum_{n|q} n^{3\nu_2} S(0, h; qn^{-1})\sigma_{2-3\nu_1-3\nu_2}(n) \cdot \Gamma^{-1}\left(\frac{3\nu_2}{2}\right)\Gamma^{-1}\left(\frac{-3\nu_1+2}{2}\right);$$

IV) The residue of $L_0(\bar{h}, q, w_2)$ at $w_2 = -\nu_1 - 2\nu_2 - 1$ is:

$$\frac{-2\pi^{\frac{3}{2}-\frac{3}{2}\nu_1-3\nu_2}}{\xi(3\nu_1)}\Gamma^{-1}\left(\frac{\alpha-\beta}{2}\right)\Gamma^{-1}\left(\frac{\alpha-\gamma}{2}\right)\Gamma^{-1}\left(\frac{w_2-\alpha}{2}\right)\Bigg|_{w_2-\alpha} = 0.$$

Similarly, the residues at $w_2 = 2\nu_1 + \nu_2 - 1$ and at $w_2 = \nu_2 - \nu_1$ are all 0.

Now let $\phi(x) \in C_c^\infty(0, \infty)$ be a test function and $\tilde{\phi}(s) = \int_0^\infty \phi(x)x^s \frac{dx}{x}$ its Mellin transform. By (3.5.33) and the Mellin inversion formula, for $\sigma > 3 + |\mathrm{Re}\,\alpha| + |\mathrm{Re}\,\beta| + |\mathrm{Re}\,\gamma|$, we have

$$\sum_{m>0} A_\nu(\delta, m)\left[e\left(\frac{m\bar{h}_\delta}{q_\delta}\right) + (-1)^k e\left(-\frac{m\bar{h}_\delta}{q_\delta}\right).\right]\phi(m) = \frac{1}{2\pi i}\int_{\mathrm{Re}\,w_2=\sigma} \tilde{\phi}(w_2 - k)L_k(\bar{h}, q, w_2)\,dw_2.$$

This is the key identity for the derivation of the Voronoi formula.

Moving the line of integration to $\mathrm{Re}\,w_2 = 1-\sigma$, picking up the poles at $w_2 = \nu_2+2\nu_1, -2\nu_2-\nu_1+2$ and $-\nu_1 + \nu_2 + 1$ when $k = 0$ and applying Lemma 3.5.7, we obtain the Voronoi formula for the

Eisenstein series $\mathscr{E}_{\min,\nu}(z)$ as follows:

$$\begin{aligned}
&\sum_{m>0} A_\nu(\delta,m,\nu)\left[e\left(\frac{m\bar{h}_\delta}{q_\delta}\right)+(-1)^k e\left(-\frac{m\bar{h}_\delta}{q_\delta}\right)\right]\phi(m)\\
&= \frac{1}{2\pi i}\int_{\operatorname{Re} w_2=\sigma}\tilde{\phi}(w_2-k)L_k(\bar{h},q,w_2)\,dw_2\\
&= \frac{1}{2\pi i}\int_{\operatorname{Re} w_2=1-\sigma}\tilde{\phi}(w_2-k)L_k(\bar{h},q,w_2)\,dw_2+\text{residue terms}\\
&= \text{MAIN TERM}+\text{RESIDUE TERMS},
\end{aligned} \tag{3.5.34}$$

where

$$\begin{aligned}
\text{MAIN TERM} &\overset{(3.5.32)}{=} \frac{1}{2\pi i}\int_{\operatorname{Re} w_2=1-\sigma}\tilde{\phi}(w_2-k)q^{w_1-2w_2+1+2k}(2\pi i)^{-k}G_1^{-1}(w,\nu)\Lambda_k(w,\nu)\,dw_2\\
&\overset{\text{Lemma 3.5.7}}{=} \frac{1}{2\pi i}\int_{\operatorname{Re} w_2=1-\sigma}\tilde{\phi}(w_2-k)q^{w_1-2w_2+1+2k}(2\pi i)^{-k}\\
&\qquad\cdot G_1^{-1}(w,\nu)\frac{(-2\pi i q)^k\pi^{\frac{w_1+w_2}{2}}}{q^{w_1+w_2}\Gamma\left(\frac{w_1+w_2}{2}\right)}\sum_{n|q}\sum_{m\neq 0}\frac{A_\nu(m,n)}{|n|^{2k+1-2w_2}|m|^{k+1-w_2}}\left(\frac{im}{|m|}\right)^k\\
&\qquad\cdot S(h,m;qn^{-1})G_2(w,\nu,k)\,dw_2\\
&= q^{1+3k}\pi^{-3k-\frac{3}{2}}(-i)^k\sum_{n|q}\sum_{m>0}\frac{n^{\nu_2+2\nu_1-2k-2}m^{2\nu_2+2\nu_1-k-2}}{\xi(3\nu_1)\xi(3\nu_2)\xi(3\nu_1+3\nu_2-1)}\\
&\qquad\cdot\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}n_1^{-3\nu_1-3\nu_2+2}n_2^{1-3\nu_1}\sigma_{1-3\nu_2,2-3\nu_1-3\nu_2}\left(\frac{n}{n_1n_2},m\right)\\
&\qquad\cdot[S(m,h;qn^{-1})+(-1)^kS(-m,h;qn^{-1})]\Phi_k^*\left(\frac{mn^2}{q^3}\right),
\end{aligned}$$

where

$$\Phi_k^*(x) = \frac{1}{2\pi i}\int_{\operatorname{Re} w=\sigma}(\pi^3x)^{-w}\frac{\Gamma\left(\frac{1+w+2k+\alpha}{2}\right)\Gamma\left(\frac{1+w+2k+\beta}{2}\right)\Gamma\left(\frac{1+w+2k+\gamma}{2}\right)}{\Gamma\left(\frac{-w-\alpha}{2}\right)\Gamma\left(\frac{-w-\beta}{2}\right)\Gamma\left(\frac{-w-\gamma}{2}\right)}\tilde{\phi}(-w-k)\,dw.$$

and the residue terms are:

$$\begin{aligned} R_1 &= \frac{\delta_{k,0}}{\xi(3\nu_2)} q^{-6\nu_1-3\nu_2+1}\pi^{3\nu_1+\frac{3}{2}\nu_2-\frac{1}{2}} \sum_{n|q} n^{3\nu_1+3\nu_2-1} S(0,h;qn^{-1})\sigma_{1-3\nu_2}(n) \\ &\qquad \cdot\left[\Gamma\left(\frac{3\nu_1}{2}\right)\right]^{-1}\left[\Gamma\left(\frac{3\nu_1+3\nu_2-1}{2}\right)\right]^{-1}\tilde{\phi}(\nu_2+2\nu_1), \end{aligned}$$

$$\begin{aligned} R_2 &= \frac{\delta_{k,0}}{\xi(3\nu_1)} q^{3\nu_1+6\nu_2-5}\pi^{-\frac{3}{2}\nu_1-3\nu_2+\frac{5}{2}} \sum_{n|q} n^{-3\nu_2+2} S(0,h;qn^{-1})\sigma_{1-3\nu_1}(n) \\ &\qquad \cdot c\left(\frac{3\nu_2}{2}\right)c\left(\frac{3\nu_1+3\nu_2-1}{2}\right)\left[\Gamma\left(\frac{-3\nu_2+2}{2}\right)\right]^{-1}\left[\Gamma\left(\frac{-3\nu_1-3\nu_2+3}{2}\right)\right]^{-1} \\ &\qquad \cdot\tilde{\phi}(-2\nu_2-\nu_1+1), \end{aligned}$$

$$\begin{aligned} R_3 &= \frac{\delta_{k,0}}{\xi(3\nu_1+3\nu_2-1)} q^{3\nu_1-3\nu_2-2}\pi^{-\frac{3\nu_1}{2}+\frac{3\nu_2}{2}+1} \sum_{n|q} n^{3\nu_2} S(0,h;qn^{-1})\sigma_{-3\nu_1-3\nu_2+2}(n) \\ &\qquad \cdot c\left(\frac{3\nu_1}{2}\right)\left[\Gamma\left(\frac{3\nu_2}{2}\right)\right]^{-1}\left[\Gamma\left(\frac{-3\nu_1+2}{2}\right)\right]^{-1}\tilde{\phi}(\nu_2-\nu_1+1), \end{aligned}$$

where

$$\delta_{k,0} = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In the case of our concern, $\nu_1 = \nu_2 = \frac{1}{3}$ and the poles are no longer simple. Yet by continuity, the result above still holds. So we take the limit $\nu_1 \to \frac{1}{3}$ and $\nu_2 \to \frac{1}{3}$.

Recall that $\delta = (h,q)$. We let $a = h/\delta$ and $c = q/\delta$, so $(a,c) = 1$, and we have

$$\text{left side of (3.5.34)} = \sum_{m>0} A_{\nu_0}(\delta,m)\left[e\left(\frac{m\bar{a}}{c}\right)+(-1)^k e\left(-\frac{m\bar{a}}{c}\right)\right]\phi(m),$$

$$\text{MAIN TERM} = (\delta c)^{1+3k}\pi^{-3k-\frac{3}{2}}(-i)^k \sum_{n|\delta c}\sum_{m>0} \frac{n^{-2k-1}m^{-k-\frac{2}{3}}}{\xi(1)^3} \cdot \sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}} \sigma_{0,0}\left(\frac{n}{n_1n_2}, m\right) \cdot [S(m, \delta a; \delta cn^{-1}) + (-1)^k S(-m, \delta a; \delta cn^{-1})]\Phi_k^*\left(\frac{mn^2}{(\delta c)^3}\right),$$

and the residue terms become:

$$R_1 = R_2 = R_3 = \frac{\delta_{k,0}}{\xi(1)}(\delta c)^{-2}\left[\pi^{\frac{1}{2}}\Gamma^{-1}\left(\frac{1}{2}\right)\right]^3 \tilde{\phi}(1)\sum_{n|\delta c} nS(0, \delta a; \delta cn^{-1})\sigma_0(n).$$

Recall that by functional equation of Riemann Zeta function, $c\left(\frac{1}{2}\right) = 1$.

Finally, by taking $k = 0$ and $k = 1$ respectively, we obtain the Voronoi formual for Eisenstein series (Proposition 3.5.2) as desired.

3.6 Proof of the main theorem

The proof of Theorem 1.2.4 is similar to that for Theorem 1.2.1. Therefore we only elaborate on the first one.

We shall first employ the Kuznetsov trace formula and the approximate functional equation to transform the left side (i.e., the spectral sum) of (1.2.1) into a sum of products of Fourier coefficients of the Eisenstein series, Kloosterman sums, and integral transforms of the test function. We shall then estimate them term by term, applying analytic tools like the Voronoi formula, etc.

Recall that the inequality we are going to prove is

$$\sum_j{}' e^{-\frac{(t_j-T)^2}{M^2}} L\left(\frac{1}{2}, \mathscr{E}_{\min,\nu_0} \times u_j\right) + \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-\frac{(t-T)^2}{M^2}} L\left(\frac{1}{2}, \mathscr{E}_{\min,\nu_0} \times E_{\frac{1}{2}+it}\right) dt \ll_\varepsilon T^{1+\varepsilon}M.$$

(For notations, see Section 1.2).

In order to be able to apply the Kuznetsov formula (c.f. Proposition 3.4.1), we insert two factors

ω_j and $\omega(t)$ and define

$$W := \sum_j{}' e^{-\frac{(t_j-T)^2}{M^2}} \omega_j L\left(\frac{1}{2}, \mathscr{E}_{\min,\nu_0} \times u_j\right) + \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-\frac{(t-T)^2}{M^2}} \omega(t) L\left(\frac{1}{2}, \mathscr{E}_{\min,\nu_0} \times E_{\frac{1}{2}+it}\right) dt$$

where

$$\begin{aligned} \omega_j &= 4\pi \left|\rho_j(1)\right|^2 / \cosh \pi t_j, \\ \omega(t) &= 4\pi \left|\rho_t^{\mathrm{Eis}}(1)\right|^2 \cosh^{-1} \pi t \end{aligned}$$

have been defined in (3.4.1).

Since $\omega_j \gg t_j^{-\varepsilon}$ and $\omega(t) \gg t^{-\varepsilon}$ for any $\varepsilon > 0$ (c.f. [24, 45]), it suffices to show that

$$W \ll_\varepsilon T^{1+\varepsilon} M.$$

To use the Kuznetsov trace formula, the test function has to be even. Therefore, we replace W by $\mathcal{W}$ defined below:

$$\mathcal{W} := \sum_j{}' k(t_j) \omega_j L\left(\frac{1}{2}, \mathscr{E}_{\min,\nu_0} \times u_j\right) + \frac{1}{4\pi}\int_{-\infty}^{\infty} k(t) \omega(t) L\left(\frac{1}{2}, \mathscr{E}_{\min,\nu_0} \times E_{\frac{1}{2}+it}\right) dt$$

where $k(t) = e^{-\frac{(t-T)^2}{M^2}} + e^{-\frac{(t+T)^2}{M^2}}$.

Applying the approximate functional equations for $L\left(\frac{1}{2}, \mathscr{E}_{\min,\nu_0} \times u_j\right)$ and $L\left(\frac{1}{2}, \mathscr{E}_{\min,\nu_0} \times E_{\frac{1}{2}+it}\right)$ (c.f. Proposition 3.3.2), we get:

$$\begin{aligned} \mathcal{W} &= 2\sum_j{}' k(t_j)\omega_j \sum_{m\geq 1}\sum_{n\geq 1} \frac{A_{\nu_0}(n,m)\lambda_j(n)}{(m^2 n)^{\frac{1}{2}}} V(m^2 n, t_j) \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty} k(t)\omega(t) \sum_{m\geq 1}\sum_{n\geq 1} \frac{A_{\nu_0}(n,m)\overline{\lambda_t^{\mathrm{Eis}}(n)}}{(m^2 n)^{\frac{1}{2}}} V(m^2 n, t)\, dt \\ &\quad + \int_{-\infty}^{\infty} R_{Eis}\left(\frac{1}{2}, t\right) dt. \end{aligned} \tag{3.6.1}$$

Here $R_{Eis}\left(\frac{1}{2}, t\right)$ is the sum of (3.3.11), (3.3.12), (3.3.13) and (3.3.14). It is easy to see that $\int_{-\infty}^{\infty} R_{Eis}\left(\frac{1}{2}, t\right)$ is convergent. It does not depend on T or M and hence can be omitted.

Using the method of smooth dyadic subdivision (partition of unity), it suffices to estimate

$$
\begin{aligned}
\mathcal{R} &:= \mathcal{W} \cdot g\left(\frac{m^2 n}{N}\right) \\
&= 2 \sum_{m \geq 1} \sum_{n \geq 1} \frac{A_{\nu_0}(n,m)}{(m^2 n)^{\frac{1}{2}}} g\left(\frac{m^2 n}{N}\right) \\
&\quad \times \left[{\sum_j}' k(t_j)\omega_j \lambda_j(n) V(m^2 n, t_j) + \frac{1}{4\pi} \int_{-\infty}^{\infty} k(t)\omega(t) \overline{\lambda_t^{\mathrm{Eis}}(n)} V(m^2 n, t)\, dt \right] \qquad (3.6.2)
\end{aligned}
$$

where g is a fixed smooth function of compact support on $[1,2]$ and $N \leq T^{3+\varepsilon}$, $\varepsilon > 0$.

Viewing $k(t)V(m^2 n, t)$ as the test function, we apply the Kuznetsov trace formula to the factor $\left[\sum_j' \ldots + \frac{1}{4\pi}\int_0^\infty \ldots dt\right]$ in $\mathcal{R}$ to transform $\mathcal{R}$ into a summation of three parts $\mathcal{R} = D + R^+ + R^-$:

- $D = \sum_{m\geq 1}\sum_{n \geq 1} \frac{A_{\nu_0}(n,m)}{(m^2n)^{\frac{1}{2}}} g\left(\frac{m^2 n}{N}\right) \delta(n,1) H_{m,n} = \sum_{m \geq 1} \frac{A_{\nu_0}(1,m)}{m} g\left(\frac{m^2}{N}\right) H_{m,1}.$ (Diagonal terms)
- $R^+ = \sum_{m\geq 1}\sum_{n \geq 1} \frac{A_{\nu_0}(n,m)}{(m^2n)^{\frac{1}{2}}} g\left(\frac{m^2 n}{N}\right) \sum_{c>0} c^{-1} S(n,1;c) H^+_{m,n}\left(\frac{4\pi\sqrt{n}}{c}\right)$
  where $H^+_{m,n}(x) = 2i\int_{-\infty}^{\infty} J_{2it}(x) \frac{V(m^2 n,t)t}{\cosh \pi t} k(t)\, dt.$
- $R^- = \sum_{m\geq 1}\sum_{n \geq 1} \frac{A_{\nu_0}(n,m)}{(m^2n)^{\frac{1}{2}}} g\left(\frac{m^2 n}{N}\right) \sum_{c>0} c^{-1} S(n,1;c) H^-_{m,n}\left(\frac{4\pi\sqrt{n}}{c}\right)$
  where $H^-_{m,n}(x) = \frac{4}{\pi}\int_{-\infty}^{\infty} K_{2it}(x) V(m^2 n, t) t \sinh \pi t k(t)\, dt.$

Now we shall estimate D, R^+ and R^- term by term.

3.6.1 Estimation for D

The estimation for D is just the same as in the case of Maass forms discussed in [34]. For completeness, we include the proof here.

Recall that by definition

$$D = \sum_{m \geq 1} \frac{A_{\nu_0}(1,m)}{m} g\left(\frac{m^2}{N}\right) H_{m,1},$$

and

$$
\begin{aligned}
H_{m,1} &= \frac{2}{\pi}\int_0^\infty \left[e^{-\frac{(t-T)^2}{M^2}} + e^{-\frac{(t+T)^2}{M^2}} \right] V(m^2, t) \tanh(\pi t) t\, dt \\
&= \frac{2}{\pi}\int_0^\infty e^{-\frac{(t-T)^2}{M^2}} V(m^2,t) \tanh(\pi t) t\, dt + O(T^{-A})
\end{aligned}
$$

with A arbitrary large. Applying Lemma 3.3.3 which gives the asymptotic behavior of $V(m^2, t)$, together with the bounds for $\sum_{n\leq N} A_{m,n}$ given in Lemma 3.6.10 and the fact that $\tanh(\pi t)$ is bounded, we get

$$D \ll_{\varepsilon,f} T^{1+\varepsilon}M.$$

3.6.2 Estimation for R^+

Recall that

$$R^+ = \sum_{m\geq 1}\sum_{n\geq 1}\frac{A_{\nu_0}(n,m)}{(m^2n)^{\frac{1}{2}}}g\left(\frac{m^2n}{N}\right)\sum_{c>0}c^{-1}S(n,1;c)H^+_{m,n}\left(\frac{4\pi\sqrt{n}}{c}\right)$$

where $H^+_{m,n}(x) = 2i\int_{-\infty}^{\infty} J_{2it}(x)\frac{V(m^2n,t)t}{\cosh\pi t}k(t)\,dt$.

We will split the summation $\sum_{c>0}$ into $\sum_{c\geq C_1/m} + \sum_{C_2/m\leq c\leq C_1/m} + \sum_{c\leq C_2/m}$ and estimate each part, where $C_1 = T$ and $C_2 = \frac{\sqrt{N}}{T^{1-\varepsilon}M}$. To be precise,

$$\begin{aligned}
R^+ &= R_1^+ + R_2^+ + R_3^+,\\
R_1^+ &= \sum_{m\geq 1}\sum_{n\geq 1}\frac{A_{\nu_0}(n,m)}{(m^2n)^{\frac{1}{2}}}g\left(\frac{m^2n}{N}\right)\sum_{c\geq C_1/m}c^{-1}S(n,1;c)H^+_{m,n}\left(\frac{4\pi\sqrt{n}}{cl}\right), && (3.6.3)\\
R_2^+ &= \sum_{m\geq 1}\sum_{n\geq 1}\frac{A_{\nu_0}(n,m)}{(m^2n)^{\frac{1}{2}}}g\left(\frac{m^2n}{N}\right)\sum_{C_2/m\leq c\leq C_1/m}c^{-1}S(n,1;c)H^+_{m,n}\left(\frac{4\pi\sqrt{n}}{c}\right), && (3.6.4)\\
R_3^+ &= \sum_{m\geq 1}\sum_{n\geq 1}\frac{A_{\nu_0}(n,m)}{(m^2n)^{\frac{1}{2}}}g\left(\frac{m^2n}{N}\right)\sum_{c\leq C_2/m}c^{-1}S(n,1;c)H^+_{m,n}\left(\frac{4\pi\sqrt{n}}{c}\right). && (3.6.5)
\end{aligned}$$

Now we will estimate each term separately.

(i) The estimation for R_1^+ exactly follows Li's method. We include it here for completeness.

We start by estimating

$$H^+_{m,n}(x) = 2i\int_{-\infty}^{\infty} J_{2it}(x)\frac{V(m^2n,t)t}{\cosh\pi t}k(t)\,dt.$$

Moving the line of integration to $\operatorname{Re} t = -100$, we get

$$H^+_{m,n}(x) = 2i\int_{-\infty}^{\infty} J_{2it+200}(x)\frac{V(m^2n,-100i+t)(-100i+t)}{\cosh\pi(-100i+t)}k(-100i+t)\,dt.$$

To bound $H^+_{m,n}(x)$, we study the bounds for $J_{2it}(x)$ and $V(m^2n,t)$.

According to the integral representation of J-Bessel function

$$J_\nu(z) = 2\frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{\pi/2}\sin^{2\nu}\theta\cos(z\cos\theta)\,d\theta \tag{3.6.6}$$

for $\operatorname{Re}\nu > -\frac{1}{2}$, we have

$$J_{2iy+200}(x) \ll \left(\frac{x}{|y|}\right)^{200} e^{\pi|y|}. \tag{3.6.7}$$

Applying Stirling's formula to (3.3.19), we get

$$V(m^2n, -100i+y) \ll \left(\frac{|y|^3}{m^2n}\right)^{100}. \tag{3.6.8}$$

Combining (3.6.6), (3.6.7) and (3.6.8), we deduce that

$$H_{m,n}^{+}(x) \ll x^{200}T^{100}(m^2n)^{-100}TM. \tag{3.6.9}$$

By Section 3.2.2 and Cauchy's inequality, we know that

$$\sum_{m^2n\le N}\left|A_{\nu_0}(m,n)\right| \ll N. \tag{3.6.10}$$

We also have Weil's bound for the Kloosterman sum

$$S(n,1;c) \ll_\varepsilon c^{\frac{1}{2}+\varepsilon}. \tag{3.6.11}$$

Recall that g is a function supported in $[1,2]$, and that $N \le T^{3+\varepsilon}$ and $T^{\frac{3}{8}+\varepsilon} \le M \le T^{\frac{1}{2}}$.

Combining these, we get

$$
\begin{aligned}
R_1^+ &\ll \sum_{N\le m^2n\le 2N} \frac{A_{\nu_0}(n,m)}{mn^{\frac{1}{2}}} \sum_{c\ge C_1/m} n^{100}c^{-200-\frac{1}{2}+\varepsilon}T^{100}(m^2n)^{-100}TM \\
&= \sum_{N\le m^2n\le 2N} \frac{A_{\nu_0}(n,m)}{mn^{\frac{1}{2}}} \sum_{c\ge C_1/m} c^{-200-\frac{1}{2}+\varepsilon}T^{100}m^{-200}TM \\
&\ll \sum_{N\le m^2n\le 2N} \frac{A_{\nu_0}(n,m)}{mn^{\frac{1}{2}}} \left(\frac{C_1}{m}\right)^{-199-\frac{1}{2}+\varepsilon} T^{100}m^{-200}TM \\
&\le \sum_{N\le m^2n\le 2N} \frac{A_{\nu_0}(n,m)}{\sqrt{N}} T^{-99-\frac{1}{2}+\varepsilon}m^{\frac{1}{2}}TM \\
&\ll T^{-99-\frac{1}{2}+\varepsilon}N^{\frac{3}{4}}TM \ll 1. \qquad (3.6.12)
\end{aligned}
$$

This concludes the estimation for R_1^+.

(ii) We will show that R_2^+ is negligible. Let us first estimate $H_{m,n}^+(x)$ for this case. We will write $H_{m,n}^+(x)$ appearing in the definition of R_3^+ as a double integral first, then apply techniques such as extending the range of integral, changing the order of integration, asymptotic expansion, stationary phase method, etc.

To estimate $H_{m,n}^+(x)$, we start with an integral representation [16]

$$
\frac{J_{2it}(x)+J_{-2it}(x)}{\cosh \pi t} = \frac{2}{\pi}\int_{-\infty}^{\infty} \sin(x\cosh\zeta)\, e\left(\frac{t\zeta}{\pi}\right) d\zeta.
$$

Plugging this into

$$
H_{m,n}^+(x) := 2i\int_{-\infty}^{\infty} \frac{J_{2it}(x)}{\cosh \pi t} V(m^2n,t)\left[e^{-\frac{(t-T)^2}{M^2}} + e^{-\frac{(t+T)^2}{M^2}}\right] t\, dt,
$$

we get

$$
\begin{aligned}
H_{m,n}^+(x) &= \frac{4i}{\pi}\int_{t=0}^{\infty}\int_{\zeta=-T^\varepsilon}^{T^\varepsilon} te^{-\frac{(t-T)^2}{M^2}} V(m^2n,t)\sin(x\cosh\zeta)e\left(\frac{t\zeta}{\pi}\right) dtd\zeta + O(T^{-A}) \\
&\overset{(\frac{t-T}{M})\mapsto t}{=} \frac{4iM}{\pi}\int_{t=-\frac{T}{M}}^{\infty}\int_{\zeta=-T^\varepsilon}^{T^\varepsilon} (T+tM)e^{-t^2}V(m^2n,tM+T)\sin(x\cosh\zeta) \\
&\qquad \cdot e\left(\frac{(tM+T)\zeta}{\pi}\right) dtd\zeta + O(T^{-A})
\end{aligned}
$$

for A arbitrarily large.

Now we split $(T + tM)$ and extend the range of integral with negligible error:

$$H^{+}_{m,n}(x) = H^{+,1}_{m,n}(x) + H^{+,2}_{m,n}(x) + O(T^{-A}),$$

where

$$\begin{aligned} H^{+,1}_{m,n}(x) &:= \frac{4iMT}{\pi}\int_{t=-\infty}^{\infty}\int_{\zeta=-T^{\varepsilon}}^{\zeta=T^{\varepsilon}} e^{-t^2}V(m^2n, tM+T)\sin(x\cosh\zeta)e\left(\frac{tM\zeta}{\pi}\right)e\left(\frac{T\zeta}{\pi}\right)dtd\zeta, \\ H^{+,2}_{m,n}(x) &:= \frac{4iM^2}{\pi}\int_{t=-\infty}^{\infty}\int_{\zeta=-T^{\varepsilon}}^{T^{\varepsilon}} te^{-t^2}V(m^2n, tM+T)\sin(x\cosh\zeta)e\left(\frac{tM\zeta}{\pi}\right)e\left(\frac{T\zeta}{\pi}\right)dtd\zeta. \end{aligned}$$

By assumption, $T^{\frac{3}{8}+\varepsilon} \leq M \leq T^{\frac{1}{2}}$. Hence it suffices to estimate $H^{+,1}_{m,n}(x)$ since $H^{+,2}_{m,n}(x)$ is a lower order term.

Define

$$k^*(t) := e^{-t^2}V(m^2n, tM+T) \tag{3.6.13}$$

and

$$\hat{k^*}(\zeta) := \int_{-\infty}^{\infty} k^*(t)e(-t\zeta)dt \tag{3.6.14}$$

to be the Fourier transform of k^*. Then

$$\begin{aligned} H^{+,1}_{m,n}(x) &= \frac{4iMT}{\pi}\int_{\zeta=-T^{\varepsilon}}^{T^{\varepsilon}} \hat{k^*}\left(-\frac{M\zeta}{\pi}\right)\sin(x\cosh\zeta)e\left(\frac{T\zeta}{\pi}\right)d\zeta \\ &\overset{\left(-\frac{M\zeta}{\pi}\mapsto\zeta\right)}{=} 4iT\int_{\zeta=-\frac{MT^{\varepsilon}}{\pi}}^{\frac{MT^{\varepsilon}}{\pi}} \hat{k^*}(\zeta)\sin\left(x\cosh\frac{\zeta\pi}{M}\right)e\left(-\frac{T\zeta}{M}\right)d\zeta. \end{aligned}$$

Since $\hat{k^*}(\zeta)$ is Fourier transform of k and hence is a Schwartz function, the integral can be extended to $(-\infty, \infty)$ with a negligible error term.

Now let

$$\begin{aligned} W_{m,n}(x) &:= T\int_{-\infty}^{\infty} \hat{k^*}(\zeta)\sin\left(x\cosh\frac{\zeta\pi}{M}\right)e\left(-\frac{T\zeta}{M}\right)d\zeta, \\ W^*_{m,n}(x) &:= T\int_{-\infty}^{\infty} \hat{k^*}(\zeta)e\left(-\frac{T\zeta}{M} - \frac{x}{2\pi}\cosh\frac{\zeta\pi}{M}\right)d\zeta. \end{aligned}$$

Then

$$W_{m,n}(x) = \frac{W^*_{m,n}(-x) - W^*_{m,n}(x)}{2i}$$

and

$$H^{+,1}_{m,n}(x) = 4iW_{m,n}(x) + O(T^{-A})$$

with A arbitrarily large.

Notice that due to the factor $e\left(\frac{-T\zeta}{M}\right)$ and the assumption $T^{3/8+\varepsilon} \le M \le T^{\frac{1}{2}}$, the contribution to $W_{m,n}(x)$ coming from $|\zeta| \ge T^{\varepsilon}$ (here $\varepsilon > 0$ arbitrarily small but fixed) is negligible. So we need only consider $|\zeta| \le T^{\varepsilon}$.

We now use the method of stationary phase. The phase ϕ in the exponential of $W^*_{m,n}(x)$ is

$$\phi(\zeta) = -\frac{T\zeta}{M} - \frac{x}{2\pi}\cosh\frac{\zeta\pi}{M},$$

so

$$\phi'(\zeta) = -\frac{T}{M} - \frac{x}{2M}\sinh\frac{\zeta\pi}{M}.$$

Then if $|x| \le T^{1-\varepsilon}M$, $W^*_{m,n}(x)$ is negligible. Therefore we may assume that $T^{1-\varepsilon}M \le |x| \le M^4$. In this case we need the asymptotic expansion of $W^*_{m,n}(x)$. By Lemma 5.1 of [31],

Proposition 3.6.1. 1) *For* $|x| \le T^{1-\varepsilon}M$ *with* $\varepsilon > 0$,

$$W^*_{m,n}(x) \ll_{\varepsilon,A} T^{-A}$$

where $A > 0$ *is arbitrarily large.*

2) *For* $T^{1-\varepsilon}M \le |x| \le M^4, T^{\frac{3}{8}+\varepsilon} \le M \le T^{\frac{1}{2}}$ *and* $L_2, L_1 \ge 1$,

$$\begin{aligned} W^*_{m,n}(x) = {} & \frac{TM}{\sqrt{|x|}} e\left(\frac{-x}{2\pi} + \frac{T^2}{\pi x}\right) \sum_{l=0}^{L_1} \sum_{0 \le l_1 \le 2l} \sum_{\frac{l_1}{4} \le l_2 \le L_2} c_{l,l_1,l_2} \frac{M^{2l-l_1}T^{4l_2-l_1}}{x^{l+3l_2-l_1}} \\ & \times \left[\hat{k}^{*(2l-l_1)}\left(\frac{-2MT}{\pi x}\right) - \frac{2\pi^6 ix}{1440M^6}(y^6\hat{k}^*(y))^{(2l-l_1)}\left(\frac{-2MT}{\pi x}\right)\right] \\ & + O\left(\frac{TM}{\sqrt{|x|}}\left(\frac{T^4}{|x|^3}\right)^{L_2+1} + T\left(\frac{M}{\sqrt{|x|}}\right)^{2L_1+3} + \frac{T|x|}{M^8}\right) \end{aligned}$$

where c_{l,l_1,l_2} are constants depending only on l, l_1 and l_2, especially $c_{0,0,0} = \frac{1+i}{\sqrt{\pi}}$.

The above proposition implies that R_2^+ is negligible [34].

(iii) Now we shall estimate R_3^+. We apply Proposition 3.6.1 again, with L_2 and L_1 chosen to be sufficiently large. The contribution to R_3^+ from the error term in (3.6.15) can be checked to be negligible [34].

To estimate the contribution to R_3^+ from the main term of (3.6.15), it suffices to consider the leading term when $l = l_1 = l_2 = 0$. Therefore we are led to estimate

$$\begin{aligned}
\tilde{R}_3^+ &:= \sum_{m\geq 1}\sum_{n\geq 1} \frac{A_{\nu_0}(n,m)}{(m^2n)^{\frac{1}{2}}} g\left(\frac{m^2n}{N}\right) \sum_{c\leq C_2/m} c^{-1} S(n,1;c) H_{m,n}^+\left(\frac{4\pi\sqrt{n}}{c}\right) \\
&= \sqrt{2} i\pi^{-1} MT e\left(-\frac{1}{8}\right) \sum_{m\geq 1}\sum_{n\geq 1} \frac{A_{\nu_0}(n,m)}{mn^{\frac{3}{4}}} g\left(\frac{m^2n}{N}\right) \\
&\qquad \cdot \sum_{c\leq C_2/m} c^{-\frac{1}{2}} S(n,1;c)\, e\left(\frac{2\sqrt{n}}{c} - \frac{T^2c}{4\pi^2\sqrt{n}}\right) \hat{k^*}\left(\frac{MTc}{2\pi^2\sqrt{n}}\right).
\end{aligned}$$

Let

$$\phi(y) = y^{-\frac{3}{4}} g\left(\frac{m^2y}{N}\right) e\left(\frac{2\sqrt{y}}{c} - \frac{T^2c}{4\pi^2\sqrt{y}}\right) \hat{k^*}\left(\frac{MTc}{2\pi^2\sqrt{y}}\right)$$

where

$$k^*(t) = e^{-t^2} V(m^2n, tM + T)$$

and

$$\hat{k^*}(\zeta) = \int_{-\infty}^{\infty} k^*(t) e(-t\zeta)\, dt$$

are defined as before.

Then

$$\begin{aligned}
\tilde{R}_3^+ &= \sqrt{2} i\pi^{-1} MT e\left(-\frac{1}{8}\right) \sum_{m\geq 1}\sum_{n\geq 1} \frac{A_{\nu_0}(n,m)}{m} \sum_{c\leq C_2/m} c^{-\frac{1}{2}} S(n,1;c)\, \phi(n) \\
&= \sqrt{2} i\pi^{-1} MT \sum_{m\geq 1}\sum_{c\leq C_2/m} c^{-\frac{1}{2}} \sum_{d\bar{d}\equiv 1 \pmod{c}} e\left(\frac{d}{c} - \frac{1}{8}\right) \sum_{n\geq 1} \frac{A_{\nu_0}(n,m)}{m} e\left(\frac{\bar{d}n}{c}\right) \phi(n).
\end{aligned}$$

Using the Voronoi formula for Eisenstein series, we get

$$\begin{aligned}
&\sum_{m>0} A_{\nu_0}(\delta, m)\, e\left(\frac{m\bar{a}}{c}\right)\phi(m) \\
&= \quad \delta c \pi^{-\frac{3}{2}} \sum_{n|\delta c} \sum_{m>0} \frac{n^{-1} m^{-\frac{2}{3}}}{\xi(1)^3} \sum_{n_1|n} \sum_{n_2|\frac{n}{n_1}} \sigma_{0,0}\left(\frac{n}{n_1 n_2}, m\right) S(m, \delta a; \delta c n^{-1}) \Phi^0_{0,1}\left(\frac{mn^2}{(\delta c)^3}\right) \\
&\quad + \delta c \pi^{-\frac{3}{2}} \sum_{n|\delta c} \sum_{m>0} \frac{n^{-1} m^{-\frac{2}{3}}}{\xi(1)^3} \sum_{n_1|n} \sum_{n_2|\frac{n}{n_1}} \sigma_{0,0}\left(\frac{n}{n_1 n_2}, m\right) S(-m, \delta a; \delta c n^{-1}) \Phi^1_{0,1}\left(\frac{mn^2}{(\delta c)^3}\right) \\
&\qquad + \text{residue terms.}
\end{aligned} \tag{3.6.15}$$

where the residue terms are

$$\frac{3\tilde{\phi}(1)}{\xi(1) m^2 c^2} \pi^{\frac{3}{2}} \Gamma^{-3}\left(\frac{1}{2}\right) \sum_{n_1|mc} n_1 S(0, md; mcn_1^{-1}) \sigma_0(m).$$

So the contribution from the residue term to $\tilde{R}_3^+$ is

$$\begin{aligned}
&3\sqrt{2\pi} i\, \Gamma^{-3}\left(\frac{1}{2}\right) \tilde{\phi}(1) \zeta^{-1}(1) MT \\
&\quad \times \sum_{m\geq 1} \frac{\sigma_0(m)}{m^2} \sum_{c\leq C_2/m} c^{-\frac{5}{2}} \sum_{d\bar{d}\equiv 1 \pmod{c}} e\left(\frac{d}{c} - \frac{1}{8}\right) \sum_{n_1|mc} n_1 S(0, md; mcn_1^{-1}).
\end{aligned} \tag{3.6.16}$$

The main term can be estimated in the same way as in [34]. Now we first estimate the contribution from the residue terms.

Since

$$\begin{aligned}
&\sum_{d\bar{d}\equiv 1 \pmod{c}} e\left(\frac{d}{c}\right) S(0, md; mcn_1^{-1}) \\
&\quad = \sum_{\substack{u \pmod{mcn_1^{-1}} \\ u\bar{u}\equiv 1 \pmod{mcn_1^{-1}}}} S(0, 1 + un_1; c) e\left(\frac{n_2 \bar{u}}{mcn_1^{-1}}\right)
\end{aligned} \tag{3.6.17}$$

and

$$S(0,a;c) = \sum_{\substack{v \pmod c \\ (v,c)=1}} e\left(\frac{av}{c}\right) \leq (a,c), \tag{3.6.18}$$

we deduce that (3.6.17) is bounded by $mc^{1+\varepsilon}$ with $\varepsilon > 0$.

Therefore we have

$$\begin{aligned}
&\sum_{m\geq 1} \frac{\sigma_0(m)}{m^2} \sum_{c\leq C_2/m} c^{-\frac{5}{2}} \sum_{d\bar{d}\equiv 1 \pmod c} e\left(\frac{d}{c}\right) \sum_{n_1|mc} n_1 S(0, md; mcn_1^{-1}) \\
&= \sum_{m\geq 1} \frac{\sigma_0(m)}{m^2} \sum_{c\leq C_2/m} c^{-\frac{5}{2}} \sum_{n_1|mc} n_1 \sum_{d\bar{d}\equiv 1 \pmod c} e\left(\frac{d}{c}\right) S(0, md; mcn_1^{-1}) \\
&\leq \sum_{m\geq 1} \frac{\sigma_0(m)}{m^2} \sum_{c\leq C_2/m} c^{-\frac{5}{2}} \sum_{n_1|mc} n_1 mc^{1+\varepsilon} \\
&= \sum_{1\leq m\leq C_2} \frac{\sigma_0(m)}{m} \sum_{c\leq C_2/m} c^{-\frac{3}{2}+\varepsilon} \sigma_0(mc).
\end{aligned} \tag{3.6.19}$$

By [1], we know that $\sigma_0(n) = o(n^\varepsilon)$ for all $\varepsilon < 0$.

Recall that

$$C_2 = \frac{\sqrt{N}}{T^{1-\varepsilon}M}, \qquad N \leq T^{3+\varepsilon}, \qquad T^{\frac{3}{8}+\varepsilon} \leq M \leq T^{\frac{1}{2}}.$$

Combining all these, we see that (3.6.19) is bounded by T^ε and therefore (3.6.16) is bounded by $T^{1+\varepsilon}M$.

3.6.3 Estimation for R^-

We will split R^- into two parts for c big and c small:

$$R_1^- = \sum_{m\geq 1}\sum_{n\geq 1} \frac{A_{\nu_0}(n,m)}{(m^2n)^{\frac{1}{2}}} g\left(\frac{m^2n}{N}\right) \sum_{c\geq C/m} c^{-1} S(n,1;c) H_{m,n}^-\left(\frac{4\pi\sqrt{n}}{c}\right) \tag{3.6.20}$$

$$R_2^- = \sum_{m\geq 1}\sum_{n\geq 1} \frac{A_{\nu_0}(n,m)}{(m^2n)^{\frac{1}{2}}} g\left(\frac{m^2n}{N}\right) \sum_{c\leq C/m} c^{-1} S(n,1;c) H_{m,n}^-\left(\frac{4\pi\sqrt{n}}{c}\right), \tag{3.6.21}$$

where

$$C = \sqrt{N} + T.$$

(i) To estimate R_1^-, we shall find the bound for $H_{m,n}^-$ first.

Recall that

$$H_{m,n}^-(x) = \frac{4}{\pi}\int_{-\infty}^{\infty} K_{2it}(x)\sinh(\pi t)k(t)V(m^2n,t)\,dt.$$

Using the formula (c.f. [47])

$$K_\nu(z) = \frac{1}{2}\pi\frac{I_{-\nu}(z)-I_\nu(z)}{\sin\nu\pi}$$

where $I_\nu(z)$ is the I-Bessel function, we can write $H_{m,n}^-$ as

$$\begin{aligned} H_{m,n}^-(x) &= 2\int_{-\infty}^{\infty}\frac{I_{-2it}(x)-I_{2it}(x)}{\sin 2it\pi}\sinh(\pi t)k(t)V(m^2n,t)t\,dt \\ &= -4\int_{-\infty}^{\infty}\frac{I_{2it}(x)}{\sin 2it\pi}\sinh(\pi t)k(t)V(m^2n,t)t\,dt. \end{aligned} \tag{3.6.22}$$

By moving the line of integration to $\operatorname{Im} t = -\sigma = -100$ we get

$$\begin{aligned} H_{m,n}^-(x) &= -4\int_{-\infty}^{\infty}\left[\sin\pi(2\sigma+2iy)\right]^{-1} I_{2\sigma+2iy}(x)\sinh\pi(-\sigma i+y) \\ &\qquad \cdot k(-\sigma i+y)V(m^2n,-\sigma i+y)(-\sigma i+y)dy. \end{aligned} \tag{3.6.23}$$

Then we can use the fact that

$$V(m^2n,-100i+y) \ll \left(\frac{|y|^2}{m^2n}\right)^{100}. \tag{3.6.24}$$

On the other hand, from the formula (c.f. [16])

$$I_\nu(x) = \frac{\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^\pi e^{x\cos\theta}\sin^{2\nu}\theta d\theta \tag{3.6.25}$$

for $\operatorname{Re}\nu > -\frac{1}{2}$, one derives that

$$I_{2\sigma+2iy}(x) \ll_\sigma x^{2\sigma}|y|^{-2\sigma}e^{\pi y}e^x.$$

Combining (3.6.23), (3.6.24) and (3.6.25), we have

$$H^{-}_{m,n}(x) \ll x^{2\sigma} e^{x} (m^2 n)^{-\sigma} T^{\sigma+1+\varepsilon} M. \tag{3.6.26}$$

Since g is compactly supported on $[1,2]$, we only need to take sum over m and n for $m^2 n \le 2N$, and hence $e^{\frac{4\pi\sqrt{n}}{c}}$ is bounded.

Plugging this into (3.6.20), and taking the trivial bound for $S(n,1;c) \le c$, we see that

$$\begin{aligned} R_1^- &\ll \sum_{m\ge1}\sum_{n\ge1} \frac{|A_{\nu_0}(n,m)|}{(m^2n)^{\frac{1}{2}}} g\left(\frac{m^2n}{N}\right) \sum_{c\ge C/m} \left(\frac{\sqrt{n}}{c}\right)^{2\sigma} T^{\sigma+1+\varepsilon}(m^2n)^{-\sigma} M \\ &\ll N^{\frac{1}{2}} T^{2-\sigma+\varepsilon} M \ll 1. \end{aligned}$$

This concludes the estimation of R_1^-.

(ii) Recall that

$$R_2^- = \sum_{m\ge1}\sum_{n\ge1} \frac{A_{\nu_0}(n,m)}{(m^2n)^{\frac{1}{2}}} g\left(\frac{m^2n}{N}\right) \sum_{c\le C/m} c^{-1} S(n,1;c) H^{-}_{m,n}\left(\frac{4\pi\sqrt{n}}{c}\right).$$

and

$$H^{-}_{m,n}(x) = \frac{4}{\pi}\int_{-\infty}^{\infty} K_{2it}(x)\sinh(\pi t) k(t) V(m^2n,t)\, dt.$$

To estimate R_2^-, we shall first estimate $H^{-}_{m,n}$. (The result will be given in Proposition 3.6.2.)

We use the following integral representation of K-Bessel function [16]:

$$K_{2it}(x) = \frac{1}{2}\cosh^{-1} t\pi \int_{-\infty}^{\infty} \cos(x\sinh\zeta) e\left(-\frac{t\zeta}{\pi}\right) d\zeta.$$

It follows that (by integration by parts in ζ once) for A arbitrarily large,

$$\begin{aligned}
H^-_{m,n}(x) &= \frac{4}{\pi}\int_0^\infty \int_{|\zeta|\le T^\varepsilon} \tanh \pi t e^{-\frac{(t-T)^2}{M^2}} V(m^2n,t) t \cos(x\sinh\zeta) \\
&\qquad \cdot e\left(-\frac{t\zeta}{\pi}\right) d\zeta dt + O(T^{-A}) \\
&\overset{(\frac{t-T}{M}\mapsto t)}{=} \frac{4M}{\pi}\int_{-\frac{T}{M}}^\infty \int_{|\zeta|\le T^\varepsilon} \tanh\pi(tM+T) e^{-t^2} V(m^2n, tM+T) \\
&\qquad \cdot(tM+T)\cos(x\sinh\zeta) e\left(-\frac{tM\zeta}{\pi}-\frac{T\zeta}{\pi}\right) dtd\zeta \\
&\qquad +O(T^{-A}).
\end{aligned}$$

Following the same trick as in the estimation of $H^+_{m,n}$, we split the factor $(tM+T)$ into two terms, and extend the t integral to $(-\infty,\infty)$ with a negligible error term, we have

$$H^-_{m,n}(x) = H^{-,1}_{m,n}(x) + H^{-,2}_{m,n}(x) + O(T^{-A}),$$

where

$$\begin{aligned}
H^{-,1}_{m,n}(x) &= \frac{4MT}{\pi}\int_{t=-\infty}^\infty \int_{|\zeta|\le T^\varepsilon} e^{-t^2} V(m^2n, tM+T)\cos(x\sinh\zeta) e\left(-\frac{(tM+T)\zeta}{\pi}\right) dtd\zeta \\
H^{-,2}_{m,n}(x) &= \frac{4M^2}{\pi}\int_{t=-\infty}^\infty \int_{|\zeta|\le T^\varepsilon} te^{-t^2} V(m^2n, tM+T)\cos(x\sinh\zeta) e\left(-\frac{(tM+T)\zeta}{\pi}\right) dtd\zeta.
\end{aligned}$$

Since $T^{\frac{3}{8}+\varepsilon} \le M \le T^{\frac{1}{2}}$, the contribution from $H^{-,2}_{m,n}$ is of lower order term, and it suffices to estimate $H^{-,1}_{m,n}$.

We adopt the same notation as in (3.6.13) and (3.6.14). One sees that

$$\begin{aligned}
H^{-,1}_{m,n}(x) &= \frac{4MT}{\pi}\int_{|\zeta|\le T^\varepsilon} \hat{k}^*\left(\frac{M\zeta}{\pi}\right)\cos(x\sinh\zeta) e\left(-\frac{T\zeta}{\pi}\right) d\zeta \\
&\overset{(\frac{M\zeta}{\pi}\mapsto\zeta)}{=} 4T\int_{|\zeta|\le \pi^{-1}MT^\varepsilon} \hat{k}^*(\zeta)\cos\left(x\sinh\frac{\zeta\pi}{M}\right) e\left(-\frac{T\zeta}{M}\right) d\zeta.
\end{aligned}$$

Since $\hat{k}^*(\zeta)$ is Fourier transform of k and hence is a Schwartz function, the integral can be extended to $(-\infty,\infty)$ with a negligible error term.

Now define

$$Y_{m,n}(x) := T\int_{-\infty}^{\infty} \hat{k}^*(\zeta)\cos\left(x\sinh\frac{\zeta\pi}{M}\right)e\left(-\frac{T\zeta}{M}\right)d\zeta$$
$$Y^*_{m,n}(x) := T\int_{-\infty}^{\infty} \hat{k}^*(\zeta)e\left(-\frac{T\zeta}{M}+\frac{x}{2\pi}\sinh\frac{\zeta\pi}{M}\right)d\zeta,$$

thus

$$Y_{m,n}(x) = \frac{Y^*_{m,n}(x)+Y^*_{m,n}(-x)}{2}$$

and

$$H^{-,1}_{m,n}(x) = 4Y_{m,n}(x) + O(T^{-A})$$

for A arbitrarily large.

Now we use the method of stationary phase again. Let

$$\Omega(\zeta) = \frac{x\sinh\frac{\zeta\pi}{M}}{2\pi} - \frac{T\zeta}{M},$$

so

$$\Omega'(\zeta) = \frac{x\cosh\frac{\zeta\pi}{M}}{2M} - \frac{T}{M}.$$

We consider the case x small, medium and large separately. Suppose $|x| \le \frac{1}{100}T$ or $|x| \ge 100T$. Then $\Omega'(\zeta) \gg \frac{T}{M} \gg T^{\varepsilon}$. By integration by parts sufficiently many times, we have

$$Y^*_{m,n}(x) \ll T^{-A}$$

with $A > 0$ arbitrarily large.

Now we are left with the case $\frac{1}{100}T \le x \le 100T$. Recall that $M \ge T^{\frac{3}{8}}$. So we have $\frac{x}{M^3} \ll T^{-\frac{1}{8}}$.

By taking the first a few terms of the Taylor series expansion of sinh, we have

$$\begin{aligned} Y^*_{m,n}(x) &= T\int_{-\infty}^{\infty}\hat{k^*}(\zeta)e\left(-\frac{T\zeta}{M}+\frac{x\zeta}{2M}+\frac{\pi^2x\zeta^3}{12M^3}+\frac{\pi^4x\zeta^5}{240M^5}\right)d\zeta \\ &\quad +O\left(T\int_{-\infty}^{\infty}|\hat{k^*}(\zeta)|\frac{|\zeta|^7|x|}{M^7}d\zeta\right). \end{aligned}$$

Again, expanding $e\left(\frac{\pi^2x\zeta^3}{12M^3}+\frac{\pi^4x\zeta^5}{240M^5}\right)$ into a Taylor series of order L_2 gives

$$\begin{aligned} Y^*_{m,n}(x) &= T\int_{-\infty}^{\infty}\hat{k^*}(\zeta)e\left(\frac{(x-2T)\zeta}{2M}\right)\sum_{l=0}^{L_2}\sum_{j=0}^{l}d_{j,l}\left(\frac{x\zeta^3}{M^3}\right)^j\left(\frac{x\zeta^5}{M^5}\right)^{l-j}d\zeta \\ &\quad +O\left(\frac{T|x|^{L_2+1}}{M^{3L_2+3}}+\frac{T|x|}{M^7}\right) \end{aligned}$$

where $d_{j,l}$ are constants coming from the Taylor expansion. Especially, $d_{0,0}=1$.

By direct computation,

$$\begin{aligned} Y^*_{m,n}(x) = T\sum_{l=0}^{L_2}\sum_{j=0}^{l}d_{j,l}\frac{x^l}{M^{5l-2j}}k^{*(5l-2j)}\left(\frac{x-2T}{2M}\right)(2\pi i)^{-5l+2j} \\ +O\left(\frac{T|x|^{L_2+1}}{M^{3L_2+3}}+\frac{T|x|}{M^7}\right). \end{aligned}$$

where $k^*(5l-2j)$ denotes the $(5l-2j)$'s derivative of k^*.

We end up with the following proposition

Proposition 3.6.2. 1) *For* $|x|\geq 100T$ *or* $x\leq\frac{1}{100}T$,

$$Y^*_{m,n}(x)\ll T^{-A}$$

where $A>0$ *is arbitrarily large and the implied constant depends only on* A.

2) *For* $\frac{1}{100}T\leq|x|\leq 100T$, $T^{\frac{3}{8}+\varepsilon}\leq M\leq T^{\frac{1}{2}}$ *and* $L_2\geq 1$,

$$\begin{aligned} Y^*_{m,n}(x) = T\sum_{l=0}^{L_2}\sum_{j=0}^{l}b_{j,l}\frac{x^l}{M^{5l-2j}}k^{*(5l-2j)}\left(\frac{x-2T}{2M}\right) \\ +O\left(\frac{T|x|^{L_2+1}}{M^{3L_2+3}}+\frac{T|x|}{M^7}\right), \end{aligned}$$

where $b_{j,l}$ are constants depending only on j and l, especially $b_{0,0} = 1$.

Remark. Recall that $H^{-,1}_{m,n} = 4Y_{m,n}(x) + O(T^{-A})$ and $Y_{m,n}(x) = \frac{1}{2}\left(Y^*_{m,n}(x) + Y^*_{m,n}(-x)\right)$. Therefore, this proposition actually finishes the estimation of $H^{-,1}_{m,n}$.

Now we return to estimate R_2^-. Recall that

$$R_2^- = \sum_{m\geq 1}\sum_{n\geq 1}\frac{A_{\nu_0}(n,m)}{(m^2n)^{\frac{1}{2}}}g\left(\frac{m^2n}{N}\right)\sum_{c\leq C/m}c^{-1}S(n,1;c)H^-_{m,n}\left(\frac{4\pi\sqrt{n}}{c}\right).$$

According to Part 1) of the proposition above, it can be seen that the contribution from $c < \frac{\sqrt{N}}{100TM}$ and $c > \frac{100\sqrt{N}}{TM}$ can be omitted.

By §3.6.10 and the trivial bound for the Kloosterman sum, one sees that the contribution R_2^- from the error term $O\left(\frac{T|x|}{M^7}\right)$ in part 2) of the above proposition is $O(T^{1+\varepsilon}M)$.

For the error term $O\left(\frac{T|x|^{L_2+1}}{M^{3L_2+3}}\right)$, we may take L_2 sufficiently large such that it becomes negligible.

Now let us estimate the contribution from the main term of $Y^*_{m,n}$ in part 2) of Proposition 3.6.2. It suffices to take the leading term $l = 0$ since all the other terms are of lower order and can be similarly handled. Thus the problem reduces to the estimation of

$$\tilde{R}_2^- := T\sum_{m\geq 1}\sum_{n\geq 1}\frac{A_{\nu_0}(n,m)}{(m^2n)^{\frac{1}{2}}}g\left(\frac{m^2n}{N}\right)\sum_{\frac{\sqrt{N}}{100Tm}\leq c\leq\frac{100\sqrt{N}}{Tm}}c^{-1}S(n,1;c)k^*\left(\frac{\frac{4\pi\sqrt{n}}{c}-2T}{2M}\right).$$

If we apply Weil's bound for the Kloosterman sum

$$S(n,1;c)\ll_\varepsilon c^{\frac{1}{2}+\varepsilon}$$

and sum over n trivially, we can only obtain

$$\tilde{R}_2^- \ll T^{\frac{1}{2}}N^{\frac{3}{4}+\varepsilon}\ll T^{\frac{11}{4}+\varepsilon},$$

which is not sufficiently small for our purpose. To improve the bound, we shall sum over n nontrivially with the help of Voronoi formula for $GL(3)$ Eisenstein series (c.f. 3.5.2).

Take

$$\phi(y) = g\left(\frac{m^2 y}{N}\right) k^*\left(\frac{\frac{4\pi\sqrt{y}}{c} - 2T}{2M}\right) y^{-\frac{1}{2}}$$

to be the test function in the Voronoi formula. Then

$$\begin{aligned}
&\sum_{n\geq 1} A_{\nu_0}(n,m) e\left(\frac{n\bar{a}}{c}\right)\phi(n) \\
&\quad = \frac{c\pi^{-\frac{5}{2}}}{4i} \sum_{n_1|cm}\sum_{n_2>0} \frac{A_{\nu_0}(n_2,n_1)}{n_1 n_2} S(ma, n_2; mcn_1^{-1}) \Phi_{0,1}^{0}\left(\frac{n_2 n_1^2}{c^3 m}\right) \\
&\quad + \frac{c\pi^{-\frac{5}{2}}}{4i} \sum_{n_1|cm}\sum_{n_2>0} \frac{A_{\nu_0}(n_2,n_1)}{n_1 n_2} S(ma, -n_2; mcn_1^{-1}) \Phi_{0,1}^{0}\left(\frac{n_2 n_1^2}{c^3 m}\right) \\
&\qquad\qquad + \text{residue terms.}
\end{aligned} \tag{3.6.27}$$

The residue term in the Voronoi formula

$$\frac{3\tilde{\phi}(1)}{\xi(1) m^2 c^2} \pi^{\frac{3}{2}} \Gamma^{-3}\left(\frac{1}{2}\right) \sum_{n_1|mc} n_1 S(0, md; mcn_1^{-1}) \sigma_0(m)$$

only involves $\tilde{\phi}(1)$, and hence does not essentially depend on the test function ϕ. Therefore, it can be estimated in the same way as the estimation for R_3^+.

To estimate the main term, it suffices to bound first term on the right side of the Voronoi formula, and we only consider $\Phi_0(x)$ since $x^{-1}\Phi_1(x)$ has similar asymptotic behavior as that of $\Phi_0(x)$. Since

$$\frac{n_2 n_1^2}{c^3 m}\frac{N}{m^2} \gg \frac{T^3}{N^{\frac{1}{2}}} \gg T^{\frac{3}{2}-\varepsilon},$$

by Lemma 2.1 in [34] for $x = \frac{n_2 n_1^2}{c^3 m}$,

$$\Phi_0(x) = 2\pi^4 xi \int_0^\infty \phi(y) \frac{d_1 \sin(6\pi x^{\frac{1}{3}} y^{\frac{1}{3}})}{(\pi^3 xy)^{\frac{1}{3}}} dy + \text{lower order terms.}$$

We will consider the large n_2 and the small n_2 separately.

If $n_2 \gg \frac{N^{\frac{1}{2}} T^{\varepsilon}}{M^3 n_1^2}$, we will have $x^{\frac{1}{3}} y^{-\frac{2}{3}} [\phi'(y)]^{-1} \gg T^{\varepsilon}$. By integration by parts many times, one shows that the contribution to $\tilde{R}_2^-$ from such terms is negligible.

So we may assume $n_2 \ll \frac{N^{\frac{1}{2}} T^{\varepsilon}}{M^3 n_1^2}$. Since $k^*(y) \ll (1+|y|)^{-A}$ for any $A > 0$, $\phi(y)$ is negligible unless

$$\left| \frac{\frac{2\pi\sqrt{y}}{c} - T}{M} \right| \leq T^{\varepsilon}.$$

This implies that

$$\frac{1}{4\pi^2}(Tc - T^{\varepsilon}Mc)^2 \leq y \leq \frac{1}{4\pi^2}(Tc + T^{\varepsilon}Mc)^2,$$

and hence

$$\Phi_0(x) \ll x^{\frac{2}{3}} \left(\frac{N}{m^2}\right)^{-\frac{5}{6}} T^{1+\varepsilon}Mc^2. \tag{3.6.28}$$

Combining (3.6.27), (3.6.27), (3.6.17) and (3.6.28), we have

$$\begin{aligned}
\tilde{R}_2^- &\ll T \sum_{m \leq \sqrt{N}} \frac{1}{m} \sum_{\frac{\sqrt{N}}{100Tm} \leq c \leq \frac{100\sqrt{N}}{Tm}} \sum_{n_1 | cm} \sum_{n_2 \ll \frac{N^{\frac{1}{2}} T^{\varepsilon}}{M^3 n_1^2}} \\
&\qquad \times \frac{|A_{\nu_0}(n_1, n_2)|}{n_1 n_2} mc^{1+\varepsilon} \left(\frac{n_2 n_1^2}{c^3 m}\right)^{\frac{2}{3}} \left(\frac{N}{m^2}\right)^{-\frac{5}{6}} T^{1+\varepsilon}Mc^2 \\
&\ll N^{\frac{1}{2}} M^{-1} T^{\varepsilon} \ll T^{1+\varepsilon} M
\end{aligned}$$

since $M \geq T^{\frac{3}{8}}$.

This concludes the estimation of R^- and hence the proof of the main theorem.

Remark.

The proof for Theorem 1.2.4 is almost the same as that for Theorem 1.2.1, so we omit it here.

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